Analysis-1 Lecture Schemes
(with Homeworks)[1]

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Preface

This work is the first member of the author's series of lecture schemes published in the Digital Library of the Faculty of Informatics. These lecture schemes are addressed to the Computer Science BSc students of Linear Algebra and of Analysis. All these works are based on the lectures and practices of the above subjects given by the author for decades in the English Course Education.

The recent work contains the topics of the first semester course Analysis-1 of the subject Analysis. It starts from the axiomatic definition of real numbers, and contains the following topics: the most important properties of real numbers, sequences, series, power series, limits of functions. It builds intensively on the following preliminary subjects:

– Mathematics in secondary school
– Discrete Mathematics
– Linear Algebra
– Precalculus Practices

This work uses the usual mathematical notations. The set of natural numbers ($\mathbb{N}$) will begin with 1. For the notation of the subset relation we will use the usual $\subseteq$ and $\subset$. The symbol $\mathbb{K}$ will denote one of the sets of real numbers ($\mathbb{R}$) or of the complex numbers ($\mathbb{C}$). Most of theorems will be supported with proof, but some of them are given without proof.

The topics are explained on a weekly basis. Every chapter contains the material of an educational week. The homework related to the topic can be found at the end of the chapter.

Thanks to my teachers and colleagues, from whom I learned a lot. I thank the lectors of this textbook – assist. prof. Dr. István Mezei and assoc. prof. Dr. Gábor Gercsák – for their thorough work and valuable advice.

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István CSÖRGŐ

1. Lesson 1

1.1. Real Numbers

In our Analysis studies we will use the natural numbers and the method of proof of mathematical induction (see: secondary school and the subject Discrete Mathematics). We will start the natural numbers from 1, that is:

$$\mathbb{N} = \{1,\ 2,\ 3,\ \ldots\}\,.$$

The real numbers and their basic properties were taught in secondary school and in Discrete Mathematics. To built up a precise analysis we need to give exactly the basic properties of real numbers in the following definition. In this connection the properties are called axioms.

1.1. Definition Let $\mathbb{R} \neq \emptyset$, and let

$$\mathbb{R} \times \mathbb{R} \ni (x,y) \mapsto x+y \quad \text{(addition), and}$$

$$\mathbb{R} \times \mathbb{R} \ni (x,y) \mapsto x \cdot y = xy \quad \text{(multiplication)}$$

be two mappings (operations), and

$$\leq \subset \mathbb{R} \times \mathbb{R}$$

be a relation (called: less or equal). Suppose that

I.
1. $\forall\,(x,y) \in \mathbb{R} \times \mathbb{R}: \quad x+y \in \mathbb{R}$ (closure under addition)
2. $\forall\,x,y \in \mathbb{R}: \quad x+y = y+x$ (commutative law).
3. $\forall\,x,y,z \in \mathbb{R}: \quad (x+y)+z = x+(y+z)$ (associative law)
4. $\exists\,0 \in \mathbb{R}\ \forall\,x \in \mathbb{R}: \quad x+0 = x$ (existence of the zero)
   It can be proved that 0 is unique. Its name is: zero.
5. $\forall\,x \in \mathbb{R}\ \exists\,(-x) \in \mathbb{R}: \quad x+(-x) = 0$. (existence of the opposite number or additive inverse)
   It can be proved that $(-x)$ is unique. Its name is: the opposite of x.

II.
1. $\forall\,(x,y) \in \mathbb{R} \times \mathbb{R}: \quad xy \in \mathbb{R}$ (closure under addition)
2. $\forall\,x,y \in \mathbb{R}: \quad xy = yx$ (commutative law).
3. $\forall\,x,y,z \in \mathbb{R}: \quad (xy)z = x(yz)$ (associative law)
4. $\exists\,1 \in \mathbb{R} \setminus \{0\}\ \forall\,x \in \mathbb{R}: \quad x \cdot 1 = x$ (existence of the unit element)
   It can be proved that 0 is unique. Its name is: unit element or simply: one.

5. $\forall\, x \in \mathbb{R} \setminus \{0\}\ \exists\, x^{-1} \in \mathbb{R}: \quad x \cdot x^{-1} = 1$. (existence of the reciprocal number or multiplicative inverse)

   It can be proved that x^{-1} is unique. Its name is: the reciprocal of x.

III. $\forall\, x, y, z \in \mathbb{R}: \quad x(y+z) = xy + xz \qquad$ (distributive law)

Using the commutativity of multiplication we obtain the other distributive law: $\forall\, x, y, z \in \mathbb{R}: \quad (x+y)z = xz + yz \quad .$

IV.
1. The relation $\leq$ is a total ordering relation (reflexive, antisymmetric, transitive, trichotomy)
2. $\forall\, x, y, z \in \mathbb{R},\ x \leq y: \quad x + z \leq y + z$
3. $\forall\, x, y, z \in \mathbb{R},\ x \leq y,\ 0 \leq z: \quad xz \leq yz$

V. (the axiom of Dedekind about the completeness)

Let $A, B \subset \mathbb{R}$, $A \neq \emptyset$, $B \neq \emptyset$ and suppose that

$$\forall\, a \in A\ \forall\, b \in B: \quad a \leq b\,.$$

Then there exists an element $s \in \mathbb{R}$ such that:

$$\forall\, a \in A\ \forall\, b \in B: \quad a \leq s \leq b\,.$$

s is called a separator element between A and B. Thus this axiom guarantees a separator element between any two nonempty sets one of them is left from the other.

In this case we say that $\mathbb{R}$ is the structure of real numbers with the two given operations (addition and multiplication) and relation. The elements of $\mathbb{R}$ are called real numbers. The above written requirements are the axioms of real numbers.

1.2. Remarks.

1. The axioms in I., II., III. express that $(\mathbb{R}, +, \cdot)$ is a field. This is the reason that the real number set $\mathbb{R}$ is often called real number field.
2. The axioms in I., II., III., IV. express that $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field.
3. There exists a (essentially unique) model for $\mathbb{R}$. This model can be constructed starting from the set theory.
4. Applying several times the associative laws of addition and multiplication we can define the sums or products of several numbers:

$$x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i \qquad (n \in \mathbb{N},\ x_i \in \mathbb{R})\,,$$

$$x_1 \cdot x_2 \cdot \cdots \cdot x_n = \prod_{i=1}^{n} x_i \qquad (n \in \mathbb{N},\ x_i \in \mathbb{R})\,.$$

Moreover – using also the commutative laws – we can define the sums and products of type

$$\sum_{i\in\Gamma} x_i \qquad \text{and} \qquad \prod_{i\in\Gamma} x_i\,,$$

where Γ is a nonempty finite index set, and $x_i \in \mathbb{R} \quad (i \in \Gamma)$.

5. We can define the subtraction as

$$x - y := x + (-y) \qquad\qquad (x, y \in \mathbb{R})$$

and the division as

$$\frac{x}{y} := x \cdot y^{-1} \qquad\qquad (x, y \in \mathbb{R},\ y \neq 0).$$

In this connection the reciprocal of x can be written as $\dfrac{1}{x}$.

6. We can define the raising to natural powers as

$$x^n := \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\ times} \qquad\qquad (x \in \mathbb{R},\ n \in \mathbb{N})\,,$$

and the raising to negative integer powers as

$$x^{-n} := \frac{1}{x^n} \qquad\qquad (x \in \mathbb{R} \setminus \{0\},\ n \in \mathbb{N})\,,$$

and the raising to zero power as

$$x^0 := 1 \qquad\qquad (x \in \mathbb{R} \setminus \{0\})\,.$$

7. We will often use the well-known identity

$$\begin{aligned} a^n - b^n &= (a-b) \cdot (a^{n-1} + a^{n-2}b + a^{n-3}b^2 \ \ldots,\ + b^{n-1}) = \\ &= (a-b) \cdot \sum_{i=0}^{n-1} a^{n-1-i} \cdot b^i \qquad (a, b \in \mathbb{R};\ n \in \mathbb{N})\,. \end{aligned} \tag{1.1}$$

8. We can define the factorial and the binomial coefficients as we have learnt in secondary school and in Discrete Mathematics:

$$n! := 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = \prod_{i=1}^{n} i \qquad\qquad (n \in \mathbb{N}), \qquad\qquad 0! := 1$$

$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k!} \qquad (n \in \mathbb{N},\ k = 0, 1, \ldots\, n)\,.$$

We will use the well-known

1.3. Theorem *[Binomial Theorem]*

For any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ holds

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n =$$
$$= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

The natural numbers (see: Discrete Mathematics) can be identified with the following elements of $\mathbb{R}$:

The natural number 1 is identified with the unit element of the multiplication guaranteed in axiom II./4.

The natural number 2 is identified with $1+1$, the natural number 3 is identified with $2+1$, the natural number 4 is identified with $3+1$, etc.

Thus the set of natural numbers is identified with the following subset of $\mathbb{R}$:

$$\{n \cdot 1 := \underbrace{1+1+\ldots+1}_{n \text{ times}} \mid n \in \mathbb{N}\}\,.$$

In this sense: $\mathbb{N} \subset \mathbb{R}$. It can be proved that each natural number is positive.

Starting out from the natural numbers we can define the well-known special number sets as follows:

The set of integers: $\mathbb{Z} := \{m - n \in \mathbb{R} \mid m, n \in \mathbb{N}\} = N \cup (-\mathbb{N}) \cup \{0\}$,

where $-\mathbb{N}$ denotes the set of the opposites of natural numbers: $-\mathbb{N} := \{-n \mid n \in \mathbb{N}\}$. The elements of $-\mathbb{N}$ are called negative integers.

The set of rational numbers: $\mathbb{Q} := \{\frac{p}{q} \in \mathbb{R} \mid p, q \in \mathbb{Z},\ q \neq 0\}$,

The set of irrational numbers: $\mathbb{R} \setminus \mathbb{Q}$.

1.4. Remark. The set of rational numbers (with the usual operations and ordering relation) satisfies all the axioms of real numbers except V. Namely, it can be shown that, for example, the following rational number sets have no rational separator element:

$$A = \{r \in \mathbb{Q} \mid r > 0,\ r^2 < 2\} \qquad B = \{r \in \mathbb{Q} \mid r > 0,\ r^2 > 2\}\,.$$

Starting out from the $\leq$ relation we can define the $<$, $\geq$, $>$ relations too. The number $x \in \mathbb{R}$ is called

- positive if $x > 0$. The set of positive real numbers is denoted by $\mathbb{R}^+$;
- negative if $x < 0$. The set of negative real numbers is denoted by $\mathbb{R}^-$;
- nonnegative if $x \geq 0$ The set of nonnegative real numbers is denoted by $\mathbb{R}_0^+$;
- non positive if $x \leq 0$. The set of non positive real numbers is denoted by $\mathbb{R}_0^-$.

Regarding the above operations and the relations $\leq$, $<$, $\geq$, $>$, all the properties and identities can be proved that we have learnt in secondary school and in the subject Discrete Mathematics.

1.5. Definition The absolute value of a real number $x \in \mathbb{R}$ is denoted by $|x|$ and it is defined as follows:

$$|x| := \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x \leq 0. \end{cases}$$

If you consider the real number line, then the absolute value means the distance between the numbers x and 0. From here we obtain intuitively that the distance between the numbers x and y is $|x-y|$. Really, denote by $d(x,y)$ the distance between x and y. Since the shifting does not affect the distance, then

$$d(x,y) = d(x-y,\, y-y) = d(x-y,\, 0) = |\, x-y|\, .$$

Sometimes it is useful to expand the set of real numbers with the ideal elements $-\infty$ and $+\infty$:

1.6. Definition The set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty,\, +\infty\}$ is called the extended real number field. We require from the ideal elements $-\infty$ and $+\infty$ the following axiom:

$$\forall\, x \in \mathbb{R} : \qquad -\infty < x < +\infty\, .$$

Thus the ordering relation is extended from $\mathbb{R}$ into $\overline{\mathbb{R}}$. Later, at the limit of sequences we will extend the algebraic operations too.

At the end of this section we define the intervals of different types:

1.7. Definition Let $a,\, b \in \overline{\mathbb{R}},\ a < b$.

- Suppose that $a, b \in \mathbb{R}$. Then $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$: closed interval;
- Suppose that $a \in \mathbb{R}$. Then $[a,b) := [a,b[:= \{x \in \mathbb{R} \mid a \leq x < b\}$: interval closed from the left, open from the right;
- Suppose that $b \in \mathbb{R}$. Then $(a,b] :=]a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$: interval open from the left, closed from the right;
- $(a,b) :=]a,b[:= \{x \in \mathbb{R} \mid a < x < b\}$: open interval.

a is called the beginning point (or: left endpoint), b is called the terminal point (or: right endpoint) of the interval.

1.8. Remark. Obviously:

$[a,+\infty) = \{x \in \mathbb{R} \mid a \leq x\}, \qquad (a,+\infty) = \{x \in \mathbb{R} \mid a < x\}, \qquad (-\infty,\, +\infty) = \mathbb{R}, \qquad$ etc.

1.2. Boundedness

1.9. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and K, $L \in \mathbb{R}$. We say that

a) K is an upper bound of H if $\forall\, x \in H : \quad x \leq K$,

b) L is a lower bound of H if $\forall\, x \in H : \quad x \geq L$.

1.10. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$. We say that

a) H is bounded above if it has an upper bound, that is $\exists\, K \in \mathbb{R}\, \forall\, x \in H : \quad x \leq K$,

b) H is bounded below if it has a lower bound, that is $\exists\, L \in \mathbb{R}\, \forall\, x \in H : \quad x \geq L$,

c) H is bounded if it is bounded above and it is bounded below.

1.11. Remark. It can be proved easily that H is bounded if and only if

$$\exists\, M > 0\ \forall\, x \in H : \quad |x| \leq M\,.$$

1.12. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say that

- a is the minimal element (or: least element) of H if $a \in H$ and $\forall\, x \in H : \quad x \geq a$. Notation: $a = \min H$.

- a is the maximal element (or: greatest element) of H if $a \in H$ and $\forall\, x \in H$: $x \leq a$. Notation: $a = \max H$.

  It can be proved that the minimal element (if it exists) is unique and that the maximal element (if it exists) is unique.

It follows from the definition that $\min H$ is the lower bound of H contained in H. Similarly, $\max H$ is the upper bound of H contained in H.

1.13. Theorem *[the Existence of the Least Upper Bound]*
Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is bounded above. Then the set of its upper bounds

$$B := \{K \in \mathbb{R} \mid K \text{ is upper bound of } H\}$$

has minimal element. This minimal element is called the least upper bound of H and is denoted by $\sup H$ *or* $\operatorname{lub} H$. *So*

$$\sup H = \operatorname{lub} H := \min B\,.$$

The term sup *is from Latin* supremum.

Proof. Let $A := H$ and B as defined in the theorem. Then A and B satisfy the assumptions of the Dedekind axiom. Thus there exists a separator element between A and B:

$$\exists\, s \in \mathbb{R}\ \forall\, a \in A\ \forall\, b \in B : \quad a \leq s \leq b\,.$$

We will show that $s = \min B$. Using the definitions of A and B we have for this s that

$$\forall\, x \in H\ \forall\, K \in B : \quad x \leq s \leq K\,.$$

The inequality $x \leq s$ $(x \in H)$ shows us that s is an upper bound of H, thus $s \in B$. This shows together with the other inequality $s \leq K$ $(K \in B)$ that $s = \min B$. $\square$

1.14. Remarks.

1. If $\alpha \in \mathbb{R}$ and we want to prove that $\sup H = \alpha$, then we make the following steps:

Step 1: Show that α is an upper bound of H, that is: $\forall\, x \in H : \quad x \leq \alpha$;

Step 2: Show that for any $\varepsilon > 0$ the number $\alpha - \varepsilon$ is not upper bound of H, that is:

$$\forall\, \varepsilon > 0\ \exists\, x \in H : \quad x > \alpha - \varepsilon\,.$$

2. $\exists\, maxH \Leftrightarrow \sup H \in H$. In this case $\sup H = \max H$.

A similar theorem can be proved about the greatest lower bound.

1.15. Theorem *[the Existence of the Greatest Lower Bound]*
Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is bounded below. Then the set of its lower bounds

$$A := \{K \in \mathbb{R} \mid K \text{ is lower bound of } H\}$$

has maximal element. This maximal element is called the greatest lower bound of H and is denoted by $\inf H$ *or* $\operatorname{glb} H$. *So*

$$\inf H = \operatorname{glb} H := \max A\,.$$

The term inf *is from Latin* infimum.

1.16. Remarks.

1. If $\alpha \in \mathbb{R}$ and we want to prove that $\inf H = \alpha$, then we make the following steps:

Step 1: Show that α is a lower bound of H, that is: $\forall\, x \in H : \quad x \geq \alpha$;

Step 2: Show that for any $\varepsilon > 0$ the number $\alpha + \varepsilon$ is not lower bound of H, that is:

$$\forall\, \varepsilon > 0\ \exists\, x \in H : \quad x < \alpha + \varepsilon\,.$$

2. $\exists\, \min H \Leftrightarrow \inf H \in H$. In this case $\inf H = \min H$.

The concepts of the least upper bound and the greatest lower bound can be extended for unbounded sets as follows:

1.17. Definition Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is unbounded above. Then $\sup H := +\infty$.

Let $\emptyset \neq H \subseteq \mathbb{R}$ and suppose that H is unbounded below. Then $\inf H := -\infty$.

Using the concepts of the least upper bound and the greatest lower bound we can give the following characterization for intervals:

1.18. Theorem *Let $\emptyset \neq H \subseteq \mathbb{R}$. Then the following three statements are equivalent:*

1. *H is an interval*
2. *$\forall\, a, b \in H,\ a < b: \quad [a, b] \subseteq H$*
3. *$(\inf H, \sup H) \subseteq H$*

1.3. Other Operations in $\mathbb{R}$

In secondary school we have learnt about the powers, roots and logarithms. In this section we will give the precise definitions of these operations. We will tell the theorems on which these definitions are based, without proof.

The definition of the powers with integer exponents was given sooner. Now let us define the roots.

1.19. Theorem *Let $a \in \mathbb{R},\ a \geq 0,\ n \in \mathbb{N}$. Then there exists uniquely the number $x \in \mathbb{R},\ x \geq 0$ for which $x^n = a$ holds. The number x can be given as:*

$$x = \sup\{t \in \mathbb{R} \mid t \geq 0,\ t^n \leq a\}\,.$$

1.20. Definition The number x in the above theorem is called the n-th root of the number a and it is denoted by $\sqrt[n]{a}$. In the case $n = 2$ it is called square root and it is denoted by $\sqrt{a}$. Furthermore we define the odd roots of negative numbers as

$$\sqrt[2n+1]{-a} := -\sqrt[2n+1]{a} \qquad\qquad (a > 0,\ n \in \mathbb{N} \cup \{0\})\,.$$

The usual identities of roots were proved in secondary school.

Using roots we can define the powers of a positive number into rational exponent, and we can prove the usual identities in connection with them.

1.21. Definition Let $a \in \mathbb{R}, a > 0$ and $r = \dfrac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z},\ q \geq 1$. Then

$$a^r = a^{\frac{p}{q}} := \sqrt[q]{a^p}$$

Now we can define the powers of a positive number into real exponent.

1.22. Definition Let $a \in \mathbb{R},\ a > 0$ and $x \in \mathbb{R}$.

- If $a > 1$, then let
$$a^x := \sup\{a^r \mid r \in \mathbb{Q},\ r < x\},$$
- If $0 < a < 1$, then let
$$a^x := \inf\{a^r \mid r \in \mathbb{Q},\ r < x\},$$
- If $a = 1$, then let $a^x := 1$.

It can be proved that in the case $x \in \mathbb{Q}$ we obtain back the powers into rational exponent. Furthermore, the usual identities are valid for the powers with real exponent.

The definition of the logarithm is based on the concept of the least upper bound too.

1.23. Theorem *Let $a, b \in \mathbb{R},\ a \neq 1,\ b > 0$. Then there exists uniquely the number $x \in \mathbb{R}$ such that $a^x = b$. A possible formula for x can be given as follows: If $a > 1$, then*
$$x := \sup\{t \in \mathbb{R} \mid a^t < b\}\,.$$
If $0 < a < 1$, then
$$x := \sup\{t \in \mathbb{R} \mid a^t > b\}\,.$$

1.24. Definition The number x in the above theorem is called the logarithm of b with base a, and it is denoted by $\log_a b$.

The usual identities of the logarithm were proved in secondary school.

1.25. Remark. Later, in Analysis-2 we will give other equivalent definitions for a^x and for $\log_a b$.

Finally, we speak some words about the number π and about the trigonometric functions. They were defined in secondary school in geometric way, and we will use them temporarily in this level. Their precise definition will be given later.

1.4. Archimedean Ordering

A well-known intuitive property of the real numbers is that you can count with them as far as you like. For example, if you count one by one:
$$0,\ 1,\ 1+1,\ 1+1+1,\ \dots, \quad \text{that is} \quad 0\cdot 1,\ 1\cdot 1,\ 2\cdot 1,\ 3\cdot 1,\ \dots,$$
then you will get over any number. In other words the set
$$\{n \cdot 1 \mid n \in \mathbb{N}\}$$
is not bounded above. This property is called the Archimedean property of the ordering of real numbers. Since this property is not in the list of axioms, we have to prove it.

1.26. Theorem *The set* $\mathbb{N} = \{n \cdot 1 \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *is not bounded above.*

Proof. Suppose indirectly that $\mathbb{N}$ is bounded above. Then $\exists\, \alpha := \sup \mathbb{N}$.

Since $\alpha - 1 < \alpha$, then $\alpha - 1$ is not an upper bound of $\mathbb{N}$. Therefore

$$\exists\, n_0 \in \mathbb{N} : \quad n_0 > \alpha - 1 .$$

However, this implies $n_0 + 1 > \alpha$ and since $n_0 + 1 \in \mathbb{N}$, we have a contradiction with the fact α is an upper bound of $\mathbb{N}$. □

1.27. Corollary.

1. Instead of counting one by one we can count by any fixed unit. More precisely: let $x,\, y \in \mathbb{R}$ be two positive numbers. Then $\frac{y}{x}$ is not upper bound of $\mathbb{N}$, thus
$$\exists\, n \in \mathbb{N} : \qquad n > \frac{y}{x} .$$
From here follows that $nx > y$. This means that the set $\{n \cdot x \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is also not bounded above. This fact is „sharp" when x is near to 0 and y is great.

2. If $\varepsilon \in \mathbb{R},\ \varepsilon > 0$, then
$$\exists\, n \in \mathbb{N} : \qquad \frac{1}{n} < \varepsilon .$$
Really, since $\frac{1}{\varepsilon}$ is not upper bound of $\mathbb{N}$ thus
$$\exists\, n \in \mathbb{N} : \qquad n > \frac{1}{\varepsilon} .$$
After rearrangement follows that $\frac{1}{n} < \varepsilon$.

3. The set of rational numbers is everywhere dense in $\mathbb{R}$. This fact is expressed in the following theorem:

   1.28. Theorem *If $a,\, b \in \mathbb{R},\ a < b$, then $(a, b) \cap \mathbb{Q} \neq \emptyset$. In other words: every open interval contains rational number.*

   Proof.

   We will prove only the case when $0 \le a < b$. The other cases can be reduced to this case. So let us suppose that $0 \le a < b$.

   Using the previous corollary
$$\exists\, q \in \mathbb{N} : \qquad \frac{1}{q} < b - a .$$
   Then using the first corollary:
$$\exists\, n \in \mathbb{N} : \qquad n \cdot \frac{1}{q} > a .$$

Let us denote by p the least of these numbers n. We will show that $a < \dfrac{p}{q} < b$. Obviously $\dfrac{p}{q} > a$. On the other hand:

$$\frac{p}{q} = \frac{p-1+1}{q} = \frac{p-1}{q} + \frac{1}{q} \leq a + \frac{1}{q} < a + (b-a) = b\,.$$

$\square$

1.5. Homework

1. Using the axioms of real numbers prove the followings:

   a) $xy = 0 \quad \Leftrightarrow \quad x = 0$ or $y = 0$

   b) $\forall\, x \in \mathbb{R},\ x \neq 0:\ x^2 > 0$

   c) $1 > 0$

2. Prove the statement in Remark 1.11.

3. Determine (without using the concept of the limit) $\sup H$, $\inf H$, $\max H$, $\min H$ if

   $$a) \quad H = \left\{ \frac{7n-2}{2n+5} \mid n \in \mathbb{N} \right\} \qquad b) \quad H = \left\{ \frac{2^{n+2}+9}{3 \cdot 2^n + 2} \mid n \in \mathbb{N} \right\}$$

4. The diameter of a nonempty subset $H \subseteq \mathbb{R}$ is defined as the distance of its „farthest" points:

   $$\operatorname{diam} H := \sup\{|x-y| \mid x, y \in H\}\,.$$

   Prove that if H is bounded, then

   $$\operatorname{diam} H = \sup H - \inf H\,.$$

   How can this formula be generalized for unbounded sets?

5. Define the homogeneous relation $\sim$ on $\mathbb{R}$ as follows:

   $$\forall\, x, y \in \mathbb{R}: \quad x \sim y \ \Leftrightarrow\ x - y \in \mathbb{Q}\,.$$

   a) Prove that $\sim$ is an equivalence relation.

   b) Denote by $[x]$ the equivalence class of $x \in \mathbb{R}$, and by $\mathbb{R}/\!\sim$ the set of the equivalence classes:

   $$[x] = \{y \in \mathbb{R} \mid x \sim y\} \qquad \text{and} \qquad \mathbb{R}/\!\sim\ = \{\,[x] \mid x \in \mathbb{R}\}\,.$$

   Prove that for any open interval I has at least one common element with any equivalence class, that is:

   $$\forall\, I \text{ open interval and } \forall\, x \in \mathbb{R}: \quad I \cap [x] \neq \emptyset\,.$$

2. Lesson 2

2.1. Some Important Inequalities

2.1. Theorem *[Triangle Inequalities]*
For any real numbers x, $y \in \mathbb{R}$ hold

a) $|x + y| \le |x| + |y|$ *(first triangle inequality)*

b) $|x - y| \ge \Big| |x| - |y| \Big|$ *(second triangle inequality)*

Proof. From the definition of the absolute value follows that

$$-|x| \le x \le |x| \quad \text{and} \quad -|y| \le y \le |y| \,.$$

Adding these inequalities we obtain that

$$-(|x| + |y|) \le x + y \le |x| + |y| \,.$$

From here follows $|x + y| \le |x| + |y|$.

To prove part b) apply part a) with $x - y$ and y:

$$|x| = |(x - y) + y| \le |x - y| + |y| \,. \quad \text{From here follows:} \quad |x| - |y| \le |x - y| \,.$$

Similarly (change x with y) we can deduce that:

$$|y| - |x| \le |y - x| = |x - y| \,.$$

The last two inequalities imply that

$$\Big| |x| - |y| \Big| \le |x - y| \,.$$

□

Remark that – applying the first triangle inequality several times – we obtain that

$$|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n| \quad (x_1,\, x_2,\, \ldots\, x_n \in \mathbb{R}) \,.$$

2.2. Theorem *[Bernoulli's Inequality]*
Let $n \in \mathbb{N},\ h \in \mathbb{R},\ h > -1$. Then

$$(1 + h)^n \ge 1 + nh \,.$$

Proof. We prove with mathematical induction. If $n = 1$, then the statement is $(1 + h)^1 \geq 1 + 1h$, which is obviously true. Let $n \in \mathbb{N}$ be a natural number for which $(1 + h)^n \geq 1 + nh$ holds. Then

$$(1+h)^{n+1} = \underbrace{(1 + h)}_{>0} \cdot (1+h)^n \geq (1+h)\cdot(1+nh) = 1+nh+h+\underbrace{nh^2}_{\geq 0} \geq 1+nh+h = 1+(n+1)h\,.$$

□

2.3. Remark. In the case $h > 0,\ n \geq 2$ the Bernoulli inequality is a simple corollary of the Binomial Theorem, namely:

$$(1+h)^n = \binom{n}{0}1^n h^0 + \binom{n}{1}1^{n-1}h^1 + \underbrace{\binom{n}{2}1^{n-2}h^2 + \ldots + \binom{n}{n}1^0 h^n}_{>0,\text{ we leave them}} >$$

$$> \binom{n}{0} + \binom{n}{1}h = 1 + nh \quad (n \geq 2).$$

Using this idea we can construct „Bernoulli inequalities of higher degree”. Let $k \in \mathbb{N}$ be fixed and write the Binomial Theorem for $n \geq k + 1$:

$$\begin{aligned}
&(1+h)^n = \\
&= \underbrace{\binom{n}{0}1^n h^0 + \ldots + \binom{n}{k-1}1^{n-k+1}h^{k-1}}_{>0,\text{ we leave them}} + \binom{n}{k}1^{n-k}h^k + \\
&+ \underbrace{\binom{n}{k+1}1^{n-k-1}h^{k+1} + \ldots + \binom{n}{n}1^0 h^n}_{>0,\text{ we leave them}} > \\
&> \binom{n}{k}h^k = \frac{n(n-1)\ldots(n-k+1)}{k!}h^k = \frac{h^k}{k!}\cdot P(n),
\end{aligned}$$

where

$$P(n) = n(n-1)\ldots(n-k+1)$$

is a k-th degree polynomial of the variable n. For $k = 1$ we obtain that $(1 + h)^n > nh$, that is almost the „classical” Bernoulli-inequality.

In the following part we will state and prove the inequality between the arithmetic and geometric means.

2.4. Definition Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$. Then the number

$$A_n := \frac{x_1 + \ldots + x_n}{n}$$

is called the arithmetic mean of the numbers $x_1, \ldots, x_n$.

2.5. Remark. It can be easily proved that $\min\{x_1, \ldots, x_n\} \le A_n \le \max\{x_1, \ldots, x_n\}$. Moreover, if the numbers $x_1, \ldots, x_n$ are not all the same (in this case necessarily $n \ge 2$), then $\min\{x_1, \ldots, x_n\} < A_n < \max\{x_1, \ldots, x_n\}$.

2.6. Definition Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}_0^+$. Then the number

$$G_n := \sqrt[n]{x_1 \cdot \ldots \cdot x_n}$$

is called the geometric mean of the nonnegative numbers $x_1, \ldots, x_n$.

2.7. Remark. It can be easily proved that $\min\{x_1, \ldots, x_n\} \le G_n \le \max\{x_1, \ldots, x_n\}$. Moreover, if the numbers $x_1, \ldots, x_n$ are positive and are not all the same (in this case necessarily $n \ge 2$), then $\min\{x_1, \ldots, x_n\} < G_n < \max\{x_1, \ldots, x_n\}$.

2.8. Theorem *[Inequality between the Arithmetic and Geometric Means]*
Let $n \in \mathbb{N},\ n \ge 2$ and $x_1, \ldots, x_n \in \mathbb{R}^+$. Then $G_n \le A_n$, that is

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \le \frac{x_1 + \ldots + x_n}{n}\,,$$

or equivalently $G_n^n \le A_n^n$, that is:

$$x_1 \cdot \ldots \cdot x_n \le \left(\frac{x_1 + \ldots + x_n}{n}\right)^n .$$

The equality holds if and only if $x_1 = \ldots = x_n$.

Proof. It is obvious that the equality holds if $x_1 = \ldots = x_n$. We have to prove that if the numbers $x_1, \ldots, x_n$ are not all the same, then the strict inequality holds. This will be proved by mathematical induction. If $n = 2$, then the equality to be proved

$$\sqrt{x_1 x_2} < \frac{x_1 + x_2}{2}$$

is equivalent to $(x_1 - x_2)^2 > 0$. However, this is true, because of $x_1 \ne x_2$.

To deduce the statement from n to $n+1$ let us take the non-all-equal positive numbers $x_1, \ldots, x_n, x_{n+1}$. We can assume – by the symmetry of the statement – that we have denoted them in nondecreasing order

$$x_1 \le \ldots \le x_n \le x_{n+1}\,,$$

and at least in one position stands the strict inequality $<$ instead of $\le$. Denote by A_{n+1} and G_{n+1} the arithmetic and the geometric mean of the above numbers respectively. Furthermore denote by A_n and G_n the arithmetic and the geometric mean of the numbers $x_1, \ldots, x_n$ respectively. We will prove that $G_n < A_n$ implies $G_{n+1} < A_{n+1}$.

Using Remark 2.5 we obtain

- in the case $x_n < x_{n+1}$:

$$A_n \le x_n < x_{n+1}, \quad \text{that is} \quad x_{n+1} - A_n > 0\,,$$

- in the case $x_n = x_{n+1}$:

$$A_n < x_n = x_{n+1}, \quad \text{that is} \quad x_{n+1} - A_n > 0\,.$$

Thus $x_{n+1} - A_n > 0$. Using this fact we can continue as follows:

$$\begin{aligned}
G_{n+1}^{n+1} &= x_1 \cdot \ldots \cdot x_{n+1} = \\
&= (x_1 \cdot \ldots \cdot x_n) \cdot x_{n+1} \leq A_n^n \cdot x_{n+1} = A_n^{n+1} + A_n^n \cdot x_{n+1} - A_n^{n+1} = \\
&= A_n^{n+1} + (n+1) \cdot A_n^n \cdot \frac{x_{n+1} - A_n}{n+1} = \binom{n+1}{0} A_n^{n+1} + \binom{n+1}{1} A_n^n \frac{x_{n+1} - A_n}{n+1} < \\
&< \sum_{k=0}^{n+1} \binom{n+1}{k} A_n^{n+1-k} \cdot \left(\frac{x_{n+1} - A_n}{n+1} \right)^k = \\
&= \left(A_n + \frac{x_{n+1} - A_n}{n+1} \right)^{n+1} = \left(\frac{nA_n + A_n + x_{n+1} - A_n}{n+1} \right)^{n+1} = \\
&= \left(\frac{nA_n + x_{n+1}}{n+1} \right)^{n+1} = \left(\frac{x_1 + \ldots + x_n + x_{n+1}}{n+1} \right)^{n+1} = A_{n+1}^{n+1}\,.
\end{aligned}$$

Taking $n+1$-th root from this inequality we have $G_{n+1} < A_{n+1}$.

Remark that the first $\leq$ in this chain is the consequence of the inductional assumption. More precisely:

- If $x_1 = \ldots = x_n$, then $x_1 \cdot \ldots \cdot x_n = G_n^n = A_n^n$, thus $x_1 \cdot \ldots \cdot x_n \cdot x_{n+1} = A_n^n \cdot x_{n+1}$,
- If $x_1, \ldots, x_n$ are not all the same, then by the inductional assumption

$$x_1 \cdot \ldots \cdot x_n = G_n^n < A_n^n, \quad \text{thus} \quad x_1 \cdot \ldots \cdot x_n \cdot x_{n+1} < A_n^n \cdot x_{n+1}\,.$$

$\square$

2.2. Complex Numbers

A quick discussion of complex numbers was given in Linear Algebra. The precise definition of the complex numbers and their operations was made in Discrete Mathematics. The set of complex numbers will be denoted by $\mathbb{C}$. The symbol $\mathbb{K}$ will denote one of the number sets $\mathbb{R}$ or $\mathbb{C}$. This notation makes the discussion possible parallel with $\mathbb{R}$ and $\mathbb{C}$.

Because of its importance we will prove the triangle inequalities in $\mathbb{C}$.

2.9. Theorem *[Triangle Inequalities in $\mathbb{C}$]*
For any complex numbers $z, w \in \mathbb{C}$ hold

$$|z + w| \leq |z| + |w| \qquad \textit{and} \qquad |z - w| \geq \Big|\, |z| - |w| \,\Big|\,.$$

Proof. It is enough to prove the first inequality, because the second one can be deduced from the first like in the real case.

To prove the first triangle inequality, let $z = a + bi$, $w = c + di$ be the algebraic forms of z and w respectively. Then we have to prove

$$|a + bi + c + di| \leq |a + bi| + |c + di| \,.$$

After squaring both sides we obtain the equivalent inequality

$$(a + c)^2 + (b + d)^2 \leq a^2 + b^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} + c^2 + d^2 \,.$$

After ordering we have the equivalent

$$ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \,. \tag{2.1}$$

If $ac + bd \leq 0$, then the above inequality is trivially true. If $ac + bd > 0$, then the squaring is an equivalent step:

$$a^2c^2 + 2abcd + b^2d^2 \leq a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \,.$$

Ordering this inequality we obtain the equivalent

$$0 \leq (bc - ad)^2 \,,$$

which is obviously true. □

2.10. Remarks.

1. The inequality (2.1) follows immediately if we apply the Cauchy inequality (see: Linear Algebra) for the vectors (a, b) and (c, d) in the Euclidean space $\mathbb{R}^2$.

2. Applying the first triangle inequality several times, we obtain that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n| \quad (z_1,\, z_2,\, \ldots\, z_n \in \mathbb{C}) \,.$$

The following theorem is a simple consequence of the first triangle inequality. It will be important at the proofs of Theorem 7.8 and of Theorem 7.18.

2.11. Theorem *Let Γ and Δ be finite nonempty index sets, and suppose that $\Gamma \subseteq \Delta$. Let*

$$x_i \in \mathbb{K} \qquad (i \in \Delta) \,.$$

Then

$$\left| \sum_{i \in \Delta} x_i - \sum_{i \in \Gamma} x_i \right| \leq \sum_{i \in \Delta} |x_i| - \sum_{i \in \Gamma} |x_i| \,.$$

Proof. The left-hand side of the above inequality is equal to

$$\left| \sum_{i \in \Delta \setminus \Gamma} x_i \right| ,$$

furthermore the right-hand side of it is equal to

$$\sum_{i \in \Delta \setminus \Gamma} |x_i| .$$

It follows immediately from here – by the first triangle inequality –, that the left-hand side is less or equal than the right-hand side. □

2.3. Functions

The concept of the function (which is a special relation) was defined in Discrete Mathematics. In this subject the students learned about some concepts and theorems about functions. In this section we review shortly this topic.

Let A and B be nonempty sets. The set of functions ordering elements from B to elements of A is denoted by $A \to B$. The domain of a function $f \in A \to B$ is denoted by D_f, the range of f is denoted by R_f. Obviously

$$D_f \subset A \quad \text{and} \quad R_f \subset B .$$

For an $x \in D_f$ $f(x)$ denotes the element of B that is ordered to x. $f(x)$ is called the function value at x.

Thus

$$R_f = \{f(x) \in B \mid x \in D_f\} = \{y \in B \mid \exists\, x \in D_f : \ f(x) = y\} .$$

The notation $f : A \to B$ means that $f \in A \to B$ and $D_f = A$.

2.12. Definition Let $f \in A \to B$. The set

$$\{(x, f(x)) \in A \times B \mid x \in D_f\} \subseteq A \times B$$

is called the graph of the function f.

2.13. Remark. If $f \in \mathbb{R} \to \mathbb{R}$, then its graph is a subset of $\mathbb{R}^2$, that is a set of points in the plane (often a plane curve). If we put this point set into the Cartesian coordinate system, then the equation of the graph of f will be $y = f(x)$.

If we want to give a function, then we have to give:

- the type of the function, that is the sets A and B,
- the domain of the function,

- the law of correspondence associating $f(x)$ to x.

We agree that if the domain is not given, then the domain will be the maximal subset of A for which $f(x)$ is defined if x is in this subset. For example, if we give a function in this way:

$$f \in \mathbb{R} \to \mathbb{R}, \quad f(x) := \frac{1}{x},$$

then – by the above agreement – $D_f = \mathbb{R} \setminus \{0\}$.

2.14. Definition Let $f \in A \to B$ and $H \subseteq D_f$. The function

$$g : H \to B, \quad g(x) := f(x) \quad (x \in H)$$

is called the restriction of f onto H, and it is denoted by $f_{|H}$.

2.15. Definition Let $f \in A \to B$. We say that f is one-to-one (injective) if

$$\forall\, u,\, v \in D_f,\ u \neq v : \quad f(u) \neq f(v)\,.$$

2.16. Definition Let $f \in A \to B$ be a one-to-one function. Then its inverse is the following function, denoted by f^{-1}:

$$f^{-1} \in B \to A, \quad D_{f^{-1}} = R_f,$$

$$f^{-1}(y) := \text{the unique } x \in D_f \text{ for which } f(x) = y \text{ holds}\,.$$

2.17. Definition Let $g \in A \to B$ and $f \in C \to D$. Suppose that the set

$$D_{f\circ g} := \{x \in D_g \mid g(x) \in D_f\} \subseteq D_g$$

is nonempty. Then the function

$$f \circ g : D_{f\circ g} \to D, (f \circ g)(x) := f(g(x))$$

is called the composition of the functions f and g. The function g is called inner function, f is called outer function.

2.4. Polynomials

2.18. Definition A function $P : \mathbb{K} \to \mathbb{K}$ is called polynomial over $\mathbb{K}$ (or simply: polynomial) if either $P = 0$ or

$$\exists\, n \in \mathbb{N} \cup \{0\} \quad \text{and} \quad \exists\, a_0,\, \ldots,\, a_n \in \mathbb{K},\ a_n \neq 0\ :$$

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = \sum_{j=0}^{n} a_jx^j \quad (x \in \mathbb{K})\,.$$

It can be proved that for each $f \neq 0$ the numbers

$$n,\ a_0, \dots, a_n$$

are unique. The number n is called the degree of the polynomial P, and is denoted by $\deg P$. The numbers a_0, …, a_n are the coefficients of P, a_n is the main coefficient. The degree of the 0 polynomial is undefined. The set of polynomials is denoted by $\mathbb{K}[x]$ or by $\mathcal{P}$.

In the following part we give an algorithm – the Horner scheme – for the division of a polynomial by a linear polynomial.

Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{j=0}^{n} a_j x^j \tag{2.2}$$

with coefficients $a_j \in \mathbb{K}$ and let $\alpha \in \mathbb{K}$. Divide P by $x - \alpha$:

$$P(x) = (x-\alpha)\cdot(b_n x^{n-1} + b_{n-1}x^{n-2} + \cdots + b_2 x + b_1) + b_0 \tag{2.3}$$

We want to determine the coefficients $b_j \in \mathbb{K}$. The numbers $b_n, \dots b_1$ will be the coefficients of the quotient polynomial, and b_0 will be the remainder.

Substituting $x = \alpha$, it is obvious that $b_0 = P(\alpha)$, which means that b_0 is the value of the polynomial at place α.

2.19. Theorem *[Horner's Scheme]*

The coefficients $b_j \quad (j = n, n-1, \dots, 1, 0)$ *can be computed with the following recursion:*

$$b_n = a_n; \qquad b_j = \alpha \cdot b_{j+1} + a_j \quad (j = n-1, n-2, \dots 1, 0).$$

Proof.

By the equations (2.2) and (2.3) we have:

$$\sum_{j=0}^{n} a_j x^j = (x-\alpha)\cdot \sum_{j=1}^{n} b_j x^{j-1} + b_0\,. \tag{2.4}$$

Let us transform the right-hand side:

$$\begin{aligned}
&(x-\alpha)\cdot\sum_{j=1}^{n} b_j x^{j-1} + b_0 = \sum_{j=1}^{n} b_j x^j - \sum_{j=1}^{n} \alpha\cdot b_j x^{j-1} + b_0 = \\
&= \sum_{j=1}^{n} b_j x^j - \sum_{j=0}^{n-1} \alpha\cdot b_{j+1} x^j + b_0 = b_n x^n + \sum_{j=1}^{n-1} b_j x^j - \sum_{j=1}^{n-1} \alpha b_{j+1} x^j - \alpha b_1 + b_0 = \\
&= b_n x^n + \sum_{j=1}^{n-1} (b_j - \alpha b_{j+1}) x^j + b_0 - \alpha b_1 = \\
&= b_n x^n + \sum_{j=0}^{n-1} (b_j - \alpha b_{j+1}) x^j\ .
\end{aligned}$$

Then we make equal the coefficients of the same degree terms on the sides of (2.4):

$$a_n = b_n \quad \text{and} \quad a_j = b_j - \alpha \cdot b_{j+1} \quad (j = n-1, \ldots, 1, 0)\,.$$

Hence we obtain by rearrangement the recursion formulas of the theorem:

$$b_n = a_n \quad \text{and} \quad b_j = \alpha \cdot b_{j+1} + a_j \quad (j = n-1, \ldots, 1, 0)\,.$$

□

2.20. Remark. The above recursion can be made in the practice with the help of the following table (Horner's table):

	a_n	a_{n-1}	a_{n-2}	$\ldots$	a_2	a_1	a_0
α	b_n	b_{n-1}	b_{n-2}	$\ldots$	b_2	b_1	b_0

We write into the upper row the coefficients of P, then we copy the first entry of the first row into the cell under it in the second row ($b_n = a_n$).

Then we compute the entries of the second row as follows:

$$b_{n-1} = \alpha \cdot b_n + a_{n-1}, \quad b_{n-2} = \alpha \cdot b_{n-1} + a_{n-2}, \quad \ldots, \quad b_1 = \alpha \cdot b_2 + a_1, \quad b_0 = \alpha \cdot b_1 + a_0$$

2.21. Example

Let $P \in \mathbb{K}[x]$ be the following polynomial

$$P(x) = x^5 - 8x^4 + 16x^3 + 18x^2 - 81x + 54, \quad \alpha = 2\,,$$

that is divide P by $(x-2)$. Then Horner's scheme is as follows:

	1	−8	16	18	−81	54
$\alpha = 2$	1	−6	4	26	−29	−4

We can read out the result of the polynomial long division from the second row of the scheme:

$$x^5 - 8x^4 + 16x^3 + 18x^2 - 81x + 54 = (x-2) \cdot (x^4 - 6x^3 + 4x^2 + 26x - 29) - 4\,.$$

On the other hand we can establish that the value of the polynomial at 2 is equal to -4, that is $P(2) = -4$.

2.22. Definition Let $P \in \mathbb{K}[x]$ be a polynomial and $\alpha \in \mathbb{K}$. The number α is called the root (place of zero) of P if $P(\alpha) = 0$.

2.23. Remark. The determination of the roots of a polynomial is generally not an easy problem. We have learned in secondary school to determine the roots of the first degree and of the second degree polynomials. These methods can be used for real and complex polynomials too.

Using Horner's Scheme we give a necessary and sufficient condition for an α to be the root of the polynomial P.

2.24. Theorem *Let $P \in \mathbb{R}[x] \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Then α is the root of P if and only if there exists a polynomial $S \in \mathbb{R}[x]$ such that*

$$P(x) = (x-\alpha) \cdot S(x) \qquad (x \in \mathbb{R})\,. \tag{2.5}$$

In words: $P(x)$ can be divided by $x - \alpha$.

Proof. Suppose that α is the root of P, that is $P(\alpha) = 0$. Using Horner's scheme we can determine the polynomial $S \in \mathbb{R}[x]$ and the number $r \in \mathbb{R}$ such that

$$P(x) = (x-\alpha) \cdot S(x) + r \qquad (x \in \mathbb{R})\,.$$

Substituting $x = \alpha$, we obtain

$$0 = P(\alpha) = (\alpha - \alpha) \cdot S(\alpha) + r = r\,.$$

Thus $r = 0$, whence

$$P(x) = (x-\alpha) \cdot S(x) + 0 = (x-\alpha) \cdot S(x) \qquad (x \in \mathbb{R})\,.$$

Conversely, suppose (2.5), and substitute $x = \alpha$. Thus we have

$$P(\alpha) = (\alpha - \alpha) \cdot S(x) = 0\,.$$

□

2.25. Corollary. Suppose that α is the root of P. Then it can be determined (e.g. with the help of Horner's scheme) a polynomial S such that

$$P(x) = (x-\alpha) \cdot S(x) \qquad (x \in \mathbb{R})\,.$$

If α is the root of S, then it can be determined a polynomial T such that

$$S(x) = (x-\alpha) \cdot T(x)\,,$$

therefore

$$P(x) = (x-\alpha) \cdot (x-\alpha) \cdot T(x) = (x-\alpha)^2 \cdot T(x) \qquad (x \in \mathbb{R})\,.$$

This process can be continued. Suppose that we can factor out the polynomial $x - \alpha$ m times. Then in the last step we have a polynomial P_1 such that

$$P(x) = (x-\alpha)^m \cdot P_1(x) \qquad (x \in \mathbb{R}) \quad \text{and} \quad P_1(\alpha) \neq 0\,.$$

The number m is called the multiplicity of the root α.

2.5. Homework

1. Prove the statement in Remark 2.5

2. Determine whether the following functions are invertible or not. If a function is invertible, determine its inverse (domain and formula).

$$a)\quad f(x) = \frac{5x+3}{2x-4} \qquad b)\quad f(x) = \frac{5x+3}{2x-4} \quad D_f = (2,\, +\infty)$$

$$c)\quad f(x) = x^2 - 6x \qquad d)\quad f(x) = x^2 - 6x \quad D_f = [4,\, +\infty]$$

3. Determine the compositions $f \circ g$ and $g \circ f$ if it exists (domain and formula).

$$f(x) = \sqrt{3-x}, \qquad g(x) = \sqrt{x^2 - 16}\,.$$

4. Using Horner's scheme factor out $x+1$ from the following polynomials:

$$a)\quad 2x^4 - x^3 - 5x^2 + x + 3$$

$$b)\quad x^5 + 6x^4 + 2x^3 - 4x^2 + 5x + 6$$

3. Lesson 3

3.1. Sequences

3.1. Definition Let H be a nonempty set.

The functions

$$a : \mathbb{N} \to H$$

are called sequences in H. For an $n \in N$ the element $a(n) \in H$ is called the n-th term of the sequence. Its usual notation is a_n.

Some notations for the sequence a:

$$a\,; \qquad (a_n)\,; \qquad (a_n,\, n \in \mathbb{N})\,; \qquad a_n \in H\ (n \in \mathbb{N})$$

3.2. Remarks.

1. Sometimes the terms are indexed starting from a fixed $p \in \mathbb{Z}$. In this case the sequence is a function defined on the set $\{n \in \mathbb{Z} \mid n \geq p\}$.

2. A sequence can be given by a formula, e.g. $a_n := \dfrac{1}{n} \quad (n \in \mathbb{N})$ or by a recursion, e.g.
$$a_1 := 1\,, \quad a_2 := 1\,, \quad a_{n+1} := a_n + a_{n-1} \quad (n \in \mathbb{N},\, n \geq 2)\,.$$

3.3. Definition The sequence $n_k \in \mathbb{N} \quad (k \in \mathbb{N})$ is called index sequence if it is strictly monotone increasing, that is

$$\forall\, n \in N : \quad n_k < n_{k+1}\,.$$

3.4. Definition Let $a : \mathbb{N} \to H$ be a sequence and let (n_k) be an index sequence. Then the sequence

$$a_{n_k} \in H \quad (k \in \mathbb{N})$$

is called the subsequence of (a_n) (composed with the index sequence (n_k)).

3.5. Example

If $a_n = \dfrac{1}{n} \quad (n \in \mathbb{N})$ and $n_k = 2^k \quad (k \in \mathbb{N})$, then

$$a_{n_k} = a_{2^k} = \frac{1}{2^k} = \left(\frac{1}{2}\right)^k \qquad (k \in \mathbb{N})\,.$$

The type of a sequence is depending on H. Some types of sequences:

- Real number sequence if $H = \mathbb{R}$ (more generally: $H \subseteq \mathbb{R}$)
- Complex number sequence if $H = \mathbb{C}$ (more generally: $H \subseteq \mathbb{C}$)
- Vector sequence if H is a vector space (more generally: H is a subset of a vector space), for example $H = \mathbb{R}^n$.
- Function sequence if H is a set consisting of functions.
- Set sequence if H is a system of sets.

The real or complex sequences are called number sequences. We will use the common notation $a : \mathbb{N} \to \mathbb{K}$ for them.

3.6. Remark. The set of number sequences (sequences of type $\mathbb{N} \to \mathbb{K}$) is an infinite dimensional vector space over $\mathbb{K}$ with respect to the usual pointwise addition and scalar multiplication.

3.2. Convergent Number Sequences

Let us discuss the real number sequence $a_n = \dfrac{1}{n}$ $(n \in \mathbb{N})$. We feel intuitively that the terms of this sequence are arbitrarily near to the number 0 if the index n is great enough. We say that the numbers $\dfrac{1}{n}$ approach 0 or converge to 0. This impression is the base of the concept of the convergency and of the limit.

To define exactly what „near to a number" means, we need the concept of neighbourhood (or ball or environment).

3.7. Definition Let $a \in \mathbb{K}$ and $r > 0$. The neighbourhood (or ball or environment) of a with radius r is the set

$$B(a,r) := \{x \in \mathbb{K} \mid \ |x-a| < r\} \subset \mathbb{K}.$$

3.8. Remarks.

1. If $\mathbb{K} = \mathbb{R}$, then the neighbourhood $B(a,r) = \{x \in \mathbb{R} \mid \ |x-a| < r\}$ is equal to the open interval $(a-r,\, a+r)$.

2. If $\mathbb{K} = \mathbb{C}$, then the neighbourhood $B(a,r) = \{z \in \mathbb{C} \mid \ |z-a| < r\}$ is equal to the open circular disk with centre a and with radius r on the complex number plane. Really, let
$$a = u + vi \quad \text{and} \quad z = x + yi\,.$$
Then
$$|z-a| = |(x-u)+(y-v)i| = \sqrt{(x-u)^2+(y-v)^2}\,,$$
thus the inequality $|z-a| < r$ is equivalent to
$$(x-u)^2+(y-v)^2 < r^2\,,$$
which describes the above mentioned circular disk.

3. Sometimes we use the closed neighbourhoods (closed balls) defined as

$$\overline{B}(a,r) := \{x \in \mathbb{K} \mid \ |x-a| \le r\} \subset \mathbb{K}\,.$$

   Similarly, we can consider that

   - If $\mathbb{K} = \mathbb{R}$, then the closed neighbourhood $\overline{B}(a,r)$ is equal to the closed interval $[a-r,\, a+r]$.
   - If $\mathbb{K} = \mathbb{C}$, then the closed neighbourhood $\overline{B}(a,r)$ is equal to the closed circular disk with centre a and with radius r.

4. Sometimes we use the neighbourhood of $a \in \mathbb{K}$ with radius $+\infty$ as

$$B(a,+\infty) := \mathbb{K}\,.$$

An important topological property of $\mathbb{K}$ (what is called T_2-property) is that any two points can be separated by disjoint neighbourhoods. This is expressed in the following theorem.

3.9. Theorem *Let $a,\, b \in \mathbb{K}$, $a \ne b$. Then*

$$\exists\, r_1,\, r_2 > 0 : \quad B(a,r_1) \cap B(b,r_2) = \emptyset\,.$$

Proof. Let $r_1 := \dfrac{|a-b|}{2} > 0$. Then for every $x \in B(a,r_1)$ holds (using the second triangle inequality):

$$|x-b| = |x-a+a-b| = |(a-b)-(a-x)| \ge |a-b|-|a-x| > |a-b|-r_1 = |a-b|-\frac{|a-b|}{2} = \frac{|a-b|}{2}.$$

thus if $r_2 := \dfrac{|a-b|}{2} > 0$, then $|x-b| > r_2$, therefore $x \notin B(b,r_2)$. □

After these preliminaries we can formulate the definition of the convergency and of the limit.

3.10. Definition The number sequence $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ is named convergent if

$$\exists A \in \mathbb{K}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N : \quad a_n \in B(A,\,\varepsilon)\,.$$

The definition can be written using inequalities as follows:

$$\exists A \in \mathbb{K}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N : \quad |a_n - A| < \varepsilon\,.$$

A number sequence is named divergent if it is not convergent.

3.11. Theorem *The number A in the above definition is unique.*

Proof. Suppose that $A_1,\, A_2 \in \mathbb{K}$ match the above definition in the role of A, and that $A_1 \neq A_2$. Then using the T_2-property of $\mathbb{K}$ (see Theorem 3.9):

$$\exists \varepsilon_1,\, \varepsilon_2 > 0 \,: \quad B(A_1, \varepsilon_1) \cap B(A_2, \varepsilon_2) = \emptyset\,.$$

By the definition to ε_1:

$$\exists N_1 \in \mathbb{N}\; \forall n \geq N_1 \,: \quad a_n \in B(A_1, \varepsilon_1)\,.$$

Similarly, to the number ε_2

$$\exists N_2 \in \mathbb{N}\; \forall n \geq N_2 \,: \quad a_n \in B(A_2, \varepsilon_2)\,.$$

Let us take e.g. the index $N := \max\{N_1\,;\, N_2\}$. Then we obtain

$$a_N \in B(A_1, \varepsilon_1) \cap B(A_2, \varepsilon_2)\,,$$

which is a contradiction. Thus $A_1 = A_2$. □

3.12. Definition Let $a_n \in \mathbb{K} \quad (n \in \mathbb{N})$ be a convergent number sequence. The unique number A in the definition 3.10 is called the limit of the sequence (a_n), and is denoted in one of the following ways:

$$\lim a = A\,, \quad \lim a_n = A\,, \quad \lim_{n\to\infty} a_n = A\,, \quad a_n \to A \quad (n \to \infty)\,,$$

$$\lim(a_n) = A\,, \quad (a_n) \to A \quad (n \to \infty)\,.$$

We often say that a_n tends to A, or a_n tends to A if n tends to infinity.

3.13. Remarks.

1. If $a : \mathbb{N} \to \mathbb{K}$ is a number sequence and $A \in \mathbb{K}$, then $\lim\limits_{n\to\infty} a_n = A$ is equivalent to

   $$\forall \varepsilon > 0\; \exists N \in \mathbb{N}\; \forall n \geq N \,: \quad a_n \in B(A,\, \varepsilon)\,,$$

   or – using inequalities – to

   $$\forall \varepsilon > 0\; \exists N \in \mathbb{N}\; \forall n \geq N \,: \quad |a_n - A| < \varepsilon\,.$$

   The number N is called a threshold index to ε.

2. It can be easily proved that a number sequence is convergent if and only if its every subsequence is convergent. In this case the limit of the sequence is equal to the limit of its any subsequence. This fact is useful at proving the divergency of a sequence: if you find two convergent subsequences with different limits, then the sequence is divergent.

3.14. Examples

1. Let (a_n) be the constant sequence, that is

$$a_n := c \quad (n \in \mathbb{N}), \quad \text{where} \quad c \in \mathbb{K} \text{ is fixed}\,.$$

   Then (a_n) is convergent and $\lim a_n = c$. Really, let $\varepsilon > 0$. Then any $N \in \mathbb{N}$ is a good threshold index, because if $n \geq N$, then

$$|a_n - c| = |c - c| = 0 < \varepsilon\,.$$

2. Let $a_n := \dfrac{1}{n} \quad (n \in \mathbb{N})$ be the harmonic sequence.

   Then (a_n) is convergent and $\lim a_n = 0$. Really, let $\varepsilon > 0$. Then – by the Archimedean property of real numbers – there exists an $N \in \mathbb{N}$ such that $N > \dfrac{1}{\varepsilon}$. This N will be a good threshold index, because for all $n \geq N$:

$$|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon\,.$$

3. Let $a_n := (-1)^n \quad (n \in \mathbb{N})$.

   Then (a_n) is divergent, because the subsequences (a_{2k}) and (a_{2k+1}) have different limits:

$$\lim_{k\to\infty} a_{2k} = (-1)^{2k} = \lim_{k\to\infty} 1 = 1$$

$$\lim_{k\to\infty} a_{2k+1} = (-1)^{2k+1} = \lim_{k\to\infty} -1 = -1$$

3.3. Convergency and Ordering

We can prove – using a similar idea as at the proof of the uniqueness of the limit – the following theorem for real number sequences:

3.15. Theorem *Let $a_n,\, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be convergent sequences and suppose that*

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n\,.$$

Then

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad a_n < b_n\,.$$

Proof. Let $A := \lim\limits_{n\to\infty} a_n$ and $B := \lim\limits_{n\to\infty} b_n$. It is assumed that $A < B$. Let $\varepsilon := \dfrac{B-A}{2} > 0$. Then – by the definition of the limits – we have

$$\exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : \quad |a_n - A| < \frac{B-A}{2}\,,$$

and

$$\exists N_2 \in \mathbb{N}\ \forall n \geq N_2 : \quad |b_n - B| < \frac{B-A}{2}\,.$$

Using the definition of the absolute value we obtain for any $n \geq N := \max\{N_1\,;\,N_2\}$ that

$$A - \frac{B-A}{2} < a_n < A + \frac{B-A}{2} \quad \text{and} \quad B - \frac{B-A}{2} < b_n < B + \frac{B-A}{2}\,.$$

Since $A + \frac{B-A}{2} = B - \frac{B-A}{2} = \frac{A+B}{2}$, then we deduce from here that

$$a_n < \frac{A+B}{2} < b_n \qquad (n \in \mathbb{N},\ n \geq N)\,.$$

□

A simple corollary of the previous theorem is the following theorem.

3.16. Theorem *Let a_n, $b_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be convergent sequences and suppose that*

$$\exists\, N_0 \in \mathbb{N}\ \forall\, n \geq N_0 : \quad a_n \leq b_n\,. \tag{3.1}$$

Then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n\,.$$

Proof. Suppose indirectly that $\lim\limits_{n\to\infty} a_n > \lim\limits_{n\to\infty} b_n$, that is $\lim\limits_{n\to\infty} b_n < \lim\limits_{n\to\infty} a_n$. Then by the previous theorem

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N : \quad b_n < a_n\,.$$

Let $n \in \mathbb{N}$ be a number greater than $\max\{N_0,\, N\}$. For this n holds $a_n > b_n$, in contradiction with (3.1). □

3.17. Remarks.

1. If we assume a stronger condition instead of (3.1), namely

$$\exists\, N_0 \in \mathbb{N}\ \forall\, n \geq N_0 : \quad a_n < b_n\,,$$

then we cannot state the strong inequality between the limits, as the following counterexample shows:

$$a_n := 0 < \frac{1}{n} =: b_n \qquad (n \in \mathbb{N})$$

but $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

2. Applying the theorem in the case when one of the sequences is the constant 0 sequence we obtain that

   - If (a_n) is convergent and $a_n \geq 0$ $(n \geq N_0)$, then $\lim\limits_{n\to\infty} a_n \geq 0$,
   - If (a_n) is convergent and $a_n \leq 0$ $(n \geq N_0)$, then $\lim\limits_{n\to\infty} a_n \leq 0$.

3.4. Convergency and Boundedness

The boundedness of a sequence is defined as the boundedness of its range.

3.18. Definition The sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is called bounded if

$$\exists\, M > 0\ \forall\, n \in \mathbb{N}: \quad |a_n| \leq M\,.$$

The number M is called a bound of the sequence.

A number sequence is called unbounded if it is not bounded.

3.19. Remark. The set of bounded sequences is an infinite dimensional vector space over $\mathbb{K}$, which is a subspace in the vector space of all $\mathbb{N} \to \mathbb{K}$ type sequences.

3.20. Definition The sequence $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ is called

- bounded above if $\quad \exists M \in \mathbb{R}\ \forall n \in \mathbb{N}: \quad a_n \leq M$. The name of M is: upper bound
- bounded below if $\quad \exists M \in \mathbb{R}\ \forall n \in \mathbb{N}: \quad a_n \geq M$. The name of M is: lower bound

It can be easily proved that a real number sequence is bounded if and only if it is bounded above and it is bounded below.

3.21. Theorem *Every convergent number sequence is bounded.*

Proof. Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a convergent sequence and $A = \lim\limits_{n\to\infty} a_n \in \mathbb{K}$. Apply the definition of convergency with $\varepsilon = 1$:

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad |a_n - A| < 1\,.$$

Use the second triangle inequality:

$$|a_n| - |A| \leq \big|\, |a_n| - |A|\, \big| \leq |a_n - A| < 1\,,$$

from where we have after rearranging

$$|a_n| < 1 + |A| \qquad (n \geq N)\,.$$

Thus obviously

$$|a_n| \leq M \quad (n \in \mathbb{N}) \quad \text{where} \quad M := \max\{|a_1|,\, |a_2|,\, \dots,\, |a_{N-1}|,\, 1 + |A|\}\,.$$

$\square$

We remark that the reverse statement is not true. The sequence $((-1)^n)$ is bounded but divergent (see example 3.14). Later we will prove that any bounded sequence has a convergent subsequence (Bolzano-Weierstrass theorem).

3.22. Example

It follows immediately from the previous theorem that an unbounded sequence is divergent. By this reason e.g. the sequences

$$a_n := n^2 \quad (n \in \mathbb{N}) \quad \text{and} \quad b_n := (-1)^n \cdot n^2 \quad (n \in \mathbb{N})$$

are divergent.

3.5. Zero Sequences

3.23. Definition The number sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is called zero sequence if it is convergent and $\lim\limits_{n\to\infty} a_n = 0$.

We will prove five short theorems about the zero sequences. They will be useful at the discussion of operations with convergent sequences.

3.24. Theorem *[T1] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and $A \in \mathbb{K}$. Then*

$$\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty} (a_n - A) = 0\,.$$

Proof. The statement is a simple consequence of the definition of the limit and of the obvious identity

$$|a_n - A| = |(a_n - A) - 0|\,.$$

□

3.25. Theorem *[T2] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. Then*

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0\,.$$

Proof. The statement is a simple consequence of the definition of the limit and of the obvious identity

$$|a_n - 0| = ||a_n| - 0|\,.$$

□

3.26. Theorem *[T3, Majorant Principle] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and $b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Suppose that (b_n) is a zero sequence and that*

$$\exists\, N_0 \in \mathbb{N}\ \forall\, n \geq N_0 : \quad |a_n| \leq b_n\,,$$

Then (a_n) is also a zero sequence.

Proof. Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} b_n = 0$, then

$$\exists\, N_1 \in \mathbb{N}\ \forall\, n \geq N_1 : \quad b_n = |b_n - 0| < \varepsilon\,.$$

Thus for the threshold index $N := \max\{N_0, N_1\}$ holds:

$$|a_n - 0| = |a_n| \leq b_n < \varepsilon\,.$$

This means that $\lim\limits_{n\to\infty} a_n = 0$. □

3.27. Theorem *[T4, Sum] Let a_n, $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be zero sequences. Then their sum $(a_n + b_n)$ is also a zero sequence.*

Proof. Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_n = 0$, then

$$\exists\, N_1 \in \mathbb{N}\ \forall\, n \geq N_1 : \quad |a_n| = |a_n - 0| < \frac{\varepsilon}{2}\,,$$

and since $\lim\limits_{n\to\infty} b_n = 0$, then

$$\exists\, N_2 \in \mathbb{N}\ \forall\, n \geq N_2 : \quad |b_n| = |b_n - 0| < \frac{\varepsilon}{2}\,.$$

Let $N := \max\{N_1, N_2\}$. It will be a good threshold index, because – using the first triangle inequality – for any $n \geq N$ holds:

$$|(a_n + b_n) - 0| = |a_n + b_n| \leq |a_n| + |b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

This means that $\lim\limits_{n\to\infty} (a_n + b_n) = 0$. □

3.28. Theorem *[T5, Product] Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a zero sequence and $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a bounded sequence. Then their product $(a_n b_n)$ is a zero sequence.*

Proof. Let $\varepsilon > 0$. Since (b_n) is bounded, then

$$\exists\, M > 0\ \forall\, n \in \mathbb{N} : \quad |b_n| \leq M\,.$$

Since $\lim\limits_{n\to\infty} a_n = 0$, then

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N : \quad |a_n| < \frac{\varepsilon}{M}\,.$$

This N will be a good threshold index, because for any $n \geq N$ holds:

$$|(a_n b_n) - 0| = |a_n| \cdot |b_n| \leq |a_n| \cdot M < \frac{\varepsilon}{M} \cdot M = \varepsilon\,.$$

This means that $\lim\limits_{n\to\infty} (a_n b_n) = 0$. □

3.29. Remark. The set of zero sequences is an infinite dimensional vector space over $\mathbb{K}$, which is a subspace in the vector space of all $\mathbb{N} \to \mathbb{K}$ type sequences.

3.6. Homework

1. Prove by definition of the limit that

   a) $\displaystyle\lim_{n\to\infty} \frac{3n-2}{7n+5} = \frac{3}{7}$ *b)* $\displaystyle\lim_{n\to\infty} \frac{n^3 - 2n^2 + 5n + 3}{4n^3 - 23n^2 + 11n + 8} = \frac{1}{4}$

   c) $\displaystyle\lim_{n\to\infty} \frac{n^3 - 3n^2 + n - 1}{1 - 2n^3 + n} = -\frac{1}{2}$ *d)* $\displaystyle\lim_{n\to\infty} \frac{n^2 + 3n - 1}{n^3 - 7n^2 + 6n - 10} = 0$

   e) $\displaystyle\lim_{n\to\infty} \frac{n-1}{n^3 + 17n - 30} = 0$

   In each question determine a threshold index to $\varepsilon = 0,001$.

4. Lesson 4

4.1. Operations with Convergent Sequences

Using Theorems $T1, \ldots, T5$ about the zero sequences we can easily discuss the operations with convergent sequences.

4.1. Theorem *[Absolute Value]*
Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a convergent sequence. Then its absolute value sequence $(|a_n|)$ is also convergent and

$$\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n| .$$

Proof. Let $A := \lim\limits_{n\to\infty} a_n$. We have to prove that $\lim\limits_{n\to\infty} |a_n| = |A|$.
Using the second triangle inequality we have:

$$\Big| |a_n| - |A| \Big| \leq |a_n - A| .$$

Since – by $T1$ – $(a_n - A)$ is a zero sequence, then by $T3$ we have that $(|a_n| - |A|)$ is a zero sequence. Thus – once more by $T1$ – $\lim\limits_{n\to\infty} |a_n| = |A|$. □

4.2. Remark. The reverse statement is not true. For example, if $a_n = (-1)^n$ $(n \in \mathbb{N})$, then $(|a_n|)$ is convergent, but (a_n) is divergent.

4.3. Theorem *[Addition]*
Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be convergent sequences. Then their sum $(a_n + b_n)$ is also convergent and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n .$$

Proof. Let $A := \lim\limits_{n\to\infty} a_n$ and $B := \lim\limits_{n\to\infty} b_n$. We have to prove that $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$.
Since – by $T1$ – $(a_n - A)$ and $(b_n - B)$ are zero sequences, then by $T4$ the sequence

$$(a_n + b_n) - (A + B) = (a_n - A) + (b_n - B)$$

is also a zero sequence. Thus $((a_n + b_n) - (A + B))$ is a zero sequence. Using once more $T1$ it follows that

$$\lim_{n\to\infty} (a_n + b_n) = A + B .$$

□

4.4. Theorem *[Multiplication]*

Let a_n, $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be convergent sequences. Then their product $(a_n b_n)$ is also convergent and

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)\,.$$

Proof. Let $A := \lim\limits_{n\to\infty} a_n$ and $B := \lim\limits_{n\to\infty} b_n$. We have to prove that $\lim\limits_{n\to\infty} (a_n b_n) = AB$.

Let us see the following transformations:

$$a_n b_n - AB = a_n b_n - A b_n + A b_n - AB = (a_n - A) b_n + A(b_n - B)\,.$$

$(a_n - A)$ and $(b_n - B)$ are zero sequences by $T1$.

The sequences (b_n) and (A) are convergent, consequently, they are bounded. Thus – using $T5$ – the sequences $((a_n - A)b_n)$ and $(A(b_n - B))$ are zero sequences.

Using $T4$ we obtain that their sum $(a_n b_n - AB)$ is a zero sequence.

Finally, using $T1$ we have $\lim\limits_{n\to\infty} (a_n b_n) = AB$. □

4.5. Corollary. If $b_n = c,\ (n \in \mathbb{N})$ is a constant sequence, then we have

$$\lim_{n\to\infty} (c a_n) = c \cdot \lim_{n\to\infty} a_n \qquad (c \in \mathbb{K})\,.$$

Combining this result with the theorem about addition we have that if the sequences a_n, $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ are convergent, then their difference $(a_n - b_n)$ is also convergent and

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\,.$$

4.6. Corollary. If $p \in \mathbb{N}$ is a fixed positive integer exponent, then

$$\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p\,.$$

4.7. Remark. The theorems about the addition and the scalar multiplication of convergent sequences imply that the set of convergent sequences is a vector space. This is an infinite dimensional subspace in the vector space of all $\mathbb{N} \to \mathbb{K}$ type sequences.

4.8. Theorem *[Reciprocal]*

Let $b_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ be a convergent sequence. Suppose that $B := \lim\limits_{n\to\infty} b_n \neq 0$. Then

a) The sequence $\left(\dfrac{1}{b_n}\right)$ is bounded

b) The sequence $\left(\dfrac{1}{b_n}\right)$ is convergent and

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}\,.$$

Proof.

a) Using Theorem 4.1 we have that $\lim\limits_{n\to\infty} |b_n| = |B| > 0$. Applying the definition of the limit for $\varepsilon := \dfrac{|B|}{2} > 0$ we obtain:

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad \frac{|B|}{2} = |B| - \frac{|B|}{2} < |b_n| < |B| + \frac{|B|}{2} = \frac{3|B|}{2}\,. \tag{4.1}$$

Thus

$$\left|\frac{1}{b_n}\right| = \frac{1}{|b_n|} \leq \max\left\{\frac{1}{|b_1|}, \ldots, \frac{1}{|b_{N-1}|}, \frac{2}{|B|}\right\}.$$

b)

$$\frac{1}{b_n} - \frac{1}{B} = \frac{B - b_n}{b_n B} = \frac{1}{b_n} \cdot \left(-\frac{1}{B}\right) \cdot (b_n - B)\,.$$

$(b_n - B)$ is a zero sequence by $T1$.

The sequences $\left(\dfrac{1}{b_n}\right)$ and $\left(-\dfrac{1}{B}\right)$ are bounded. Thus – using $T5$ – the sequence on the right side of the above equality is a zero sequence.

Finally, using $T1$ we have $\lim\limits_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{B}$.

□

4.9. Corollary. If $p \in \mathbb{Z},\ p < 0$ is a fixed negative integer exponent, then

$$\lim_{n\to\infty} b_n^p = (\lim_{n\to\infty} b_n)^p\,.$$

Combining the theorems about the reciprocal and about the multiplication we obtain the theorem about the quotient of two sequences:

4.10. Theorem *[Division]*
Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ $b_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ be convergent sequences. Suppose that $\lim\limits_{n\to\infty} b_n \neq 0$. Then their quotient $\left(\dfrac{a_n}{b_n}\right)$ is also convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}\,.$$

Proof.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(a_n \cdot \frac{1}{b_n}\right) = (\lim_{n\to\infty} a_n) \cdot \lim_{n\to\infty} \frac{1}{b_n} = (\lim_{n\to\infty} a_n) \cdot \frac{1}{\lim\limits_{n\to\infty} b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}\,.$$

□

4.11. Remark. The theorems about the reciprocal and about the quotient and their corollaries can be extended to the case when the assumption $b_n \neq 0$ $(n \in \mathbb{N})$ is not required, only the assumption $\lim\limits_{n\to\infty} b_n \neq 0$ is required. In this case – using the estimation in (4.1) – we have:

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad |b_n| > \frac{|B|}{2} > 0\,,$$

that is

$$b_n \neq 0 \quad \text{for} \quad n = N,\, N+1,\, \ldots$$

So we can take the sequence $(b_n,\ n \in \mathbb{N},\, n \geq N)$ instead of $(b_n,\ n \in \mathbb{N})$.

4.12. Theorem *[q-th root]*

Let $q \in \mathbb{N}$, $q \geq 2$ and $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a convergent sequence. Suppose that $a_n \geq 0$ $(n \in \mathbb{N})$. Then its q-th root sequence $(\sqrt[q]{a_n})$ is also convergent, and

$$\lim_{n\to\infty} \sqrt[q]{a_n} = \sqrt[q]{\lim_{n\to\infty} a_n}\,.$$

Remark that by Theorem 3.16 and its corollary $\lim\limits_{n\to\infty} a_n \geq 0$.

Proof. Let $A := \lim\limits_{n\to\infty} a_n \geq 0$. We have to prove that $\lim\limits_{n\to\infty} \sqrt[q]{a_n} = \sqrt[q]{A}$.

First suppose $A > 0$. We will use the well-known elementary identity (see: (1.1))

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$

with $n = q$, $a = \sqrt[q]{a_n} \geq 0$, $b = \sqrt[q]{A} > 0$. Thus

$$\begin{aligned} &|a_n - A| = |(\sqrt[q]{a_n})^q - (\sqrt[q]{A})^q| = \\ &= |\sqrt[q]{a_n} - \sqrt[q]{A}| \cdot \left((\sqrt[q]{a_n})^{q-1} + (\sqrt[q]{a_n})^{q-2}\sqrt[q]{A} + \ldots + \sqrt[q]{a_n}(\sqrt[q]{A})^{q-2} + (\sqrt[q]{A})^{q-1}\right)\,. \end{aligned}$$

Here we have used that each term of the second factor on the right side is nonnegative and the last term is positive. This is the reason that the absolute value is not written around the second factor.

After leaving the first $q-1$ nonnegative terms from the second factor we obtain the following inequality:

$$|a_n - A| \geq |\sqrt[q]{a_n} - \sqrt[q]{A}| \cdot (\sqrt[q]{A})^{q-1}\,.$$

Consequently

$$|\sqrt[q]{a_n} - \sqrt[q]{A}| \leq \frac{|a_n - A|}{(\sqrt[q]{A})^{q-1}}$$

Since – by $T1$ and $T2$ – $(|a_n - A|)$ is a zero sequence, then by $T3$ we have that $(\sqrt[q]{a_n} - \sqrt[q]{A})$ is a zero sequence. Thus – once more by $T1$ – $\lim\limits_{n\to\infty} \sqrt[q]{a_n} = \sqrt[q]{A}$.

In the remainder case $A = 0$ we will use the definition of the limit. Let $\varepsilon > 0$. Then $\varepsilon^q > 0$, therefore

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad 0 \leq a_n = |a_n - 0| < \varepsilon^q\,.$$

Taking q-th root we obtain

$$\exists\, N \in \mathbb{N}\ \forall n \geq N: \quad 0 \leq |\sqrt[q]{a_n} - \sqrt[q]{0}| = \sqrt[q]{a_n} < \sqrt[q]{\varepsilon^q} = \varepsilon\,.$$

This implies the statement of the theorem in the case $A = 0$. □

4.13. Corollary. If $a_n \in \mathbb{R}$ $a_n > 0$ $(n \in \mathbb{N})$ and $r \in \mathbb{Q}$ is a fixed rational exponent, then

$$\lim_{n\to\infty} a_n^r = (\lim_{n\to\infty} a_n)^r\,.$$

4.14. Theorem *[Sandwich Theorem]*

Let a_n, b_n, $c_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be real number sequences and suppose that

a) $\exists\, N_0 \in \mathbb{N}\ \forall n \geq N_0: \quad a_n \leq b_n \leq c_n$ *and that*

b) (a_n) *and* (c_n) *are convergent and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n =: A$.

Then (b_n) *is also convergent and* $\lim\limits_{n\to\infty} b_n = A$.

Proof. Let us start from the inequalities

$$a_n \leq b_n \leq c_n \qquad (n \in \mathbb{N},\ n \geq N_0)\,.$$

After subtracting a_n we have

$$0 \leq b_n - a_n \leq c_n - a_n \qquad (n \in \mathbb{N},\ n \geq N_0)\,.$$

Since

$$\lim_{n\to\infty}(c_n - a_n) = \lim_{n\to\infty} c_n - \lim_{n\to\infty} a_n = A - A = 0\,,$$

then $(c_n - a_n)$ is a zero sequence. Using $T3$ we obtain that $(b_n - a_n)$ is also a zero sequence. Finally

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}((b_n - a_n) + a_n) = \lim_{n\to\infty}(b_n - a_n) + \lim_{n\to\infty} a_n = 0 + A = A\,.$$

□

4.2. Some Important Convergent Sequences

In the Examples 3.14 we have seen that

$$\lim_{n\to\infty} a_n = c\ (c \in \mathbb{K}) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0\,.$$

Now let us discuss the geometric sequence.

4.15. Definition Let $q \in \mathbb{K}$ be a fixed number. Then the sequence

$$a_n := q^n \quad (n \in \mathbb{N})$$

is called a geometric sequence (with base q or with quotient q).

4.16. Theorem *The geometric sequence is convergent if and only if $|q| < 1$ or $q = 1$. In this case*

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \textit{if } |q| < 1 \\ \\ 1 & \textit{if } q = 1 \end{cases}$$

Proof. The statement of the theorem is trivial if $q = 0$ or if $q = 1$. Suppose that $0 < |q| < 1$. Then $\dfrac{1}{|q|} > 1$ and – using the Bernoulli inequality (see Theorem 2.2) –

$$\frac{1}{|q|^n} = \left(\frac{1}{|q|}\right)^n = \left(1 + \frac{1}{|q|} - 1\right)^n \geq 1 + n \cdot \left(\frac{1}{|q|} - 1\right) > n \cdot \left(\frac{1}{|q|} - 1\right).$$

After rearranging we have

$$0 \leq |q^n| = |q|^n \leq \frac{1}{\dfrac{1}{|q|} - 1} \cdot \frac{1}{n} \qquad (n \in \mathbb{N}).$$

The right side sequence tends to 0. Using the Sandwich Theorem we obtain $\lim\limits_{n\to\infty} |q^n| = 0$. Using Theorem 3.25 we have $\lim\limits_{n\to\infty} q^n = 0$.

Suppose that $|q| > 1$. Once more using the Bernoulli inequality:

$$|q^n| = |q|^n = (1 + |q| - 1)^n \geq 1 + n \cdot |q| > n \cdot |q|,$$

which implies that the sequence (q^n) is unbounded. Consequently it is divergent.

Finally, suppose that $|q| = 1$ but $q \neq 1$. Suppose indirectly that $(a_n = q^n)$ is convergent and denote by A its limit. Then by Theorem 4.1 we have

$$|A| = |\lim_{n\to\infty} q^n| = \lim_{n\to\infty} |q^n| = \lim_{n\to\infty} |q|^n = \lim_{n\to\infty} 1^n = 1,$$

which implies $A \neq 0$.

On the other hand $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = A$, therefore

$$\begin{aligned} 0 = A - A &= \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} (q^{n+1} - q^n) = \\ &= \lim_{n\to\infty} q^n(q-1) = (q-1) \cdot \lim_{n\to\infty} q^n = (q-1) \cdot A. \end{aligned}$$

We obtained that

$$0 = (q-1) \cdot A,$$

which is a contradiction, because on the right side stands the product of two nonzero numbers. □

We have finished the discussion of the geometric sequence. In the following theorems we will discuss some other interesting convergent sequences.

4.17. Theorem *Let $a \in \mathbb{R}$, $a > 0$ be fixed. Then*

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

Proof. Suppose first that $a > 1$.

Let $n \geq 2$ and apply the inequality between the arithmetic and the geometric means (see Theorem 2.8) for the n pieces of non-all-equal positive numbers

$$a,\ 1,\ 1,\ \dots,\ 1\ :$$

$$1 < \sqrt[n]{a} = \sqrt[n]{a \cdot 1 \cdot \ \dots\ \cdot 1} < \frac{a+n-1}{n} = 1 + \frac{a-1}{n}\,.$$

The sequence on the right side obviously tends to 1. Applying the Sandwich Theorem we obtain the statement of the theorem.

The case $0 < a < 1$ can be reduced to the first case. Really, since $\frac{1}{a} > 1$, then

$$\sqrt[n]{a} = \frac{1}{\frac{1}{\sqrt[n]{a}}} = \frac{1}{\sqrt[n]{\frac{1}{a}}} \longrightarrow \frac{1}{1} = 1\,.$$

Finally, the case $a = 1$ is trivial. □

4.18. Theorem

$$\lim_{n\to\infty} \sqrt[n]{n} = 1\,.$$

Proof. Suppose that $n \geq 2$ and apply the inequality between the arithmetic and the geometric means (see Theorem 2.8) for the n pieces of non-all-equal positive numbers

$$\sqrt{n},\ \sqrt{n},\ 1,\ 1,\ \dots,\ 1\ :$$

$$1 < \sqrt[n]{n} = \sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot 1 \cdot\ \dots\ \cdot 1} < \frac{2\sqrt{n}+n-2}{n} = \frac{2}{\sqrt{n}} + 1 - \frac{2}{n}$$

The sequence on the right side obviously tends to 1. Applying the Sandwich Theorem we obtain the statement of the theorem. □

4.19. Corollary. For any fixed $r \in \mathbb{Q}$:

$$\lim_{n\to\infty} \sqrt[n]{n^r} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^r = 1^r = 1\,.$$

4.20. Theorem *Let $q \in \mathbb{K}$ and $k \in \mathbb{N}$ be fixed. Then*

$$\lim_{n\to\infty} n^k \cdot q^n = 0\,.$$

Proof. For any $n \in \mathbb{N}$ we have

$$|n^k q^n| = n^k \cdot |q|^n = \left(\left(\sqrt[n]{n}\right)^n\right)^k \cdot |q|^n = \left(\sqrt[n]{n^k}\cdot|q|\right)^n\,. \tag{4.2}$$

Let $s \in \mathbb{R}$, $|q| < s < 1$ be fixed. Since $\lim\limits_{n\to\infty}\left(\sqrt[n]{n^k}\right)\cdot|q| = 1\cdot|q| = |q|$, then by the definition of the limit:

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad \left(\sqrt[n]{n^k}\right)\cdot|q| < s\,.$$

Thus we can continue (4.2) for $n \geq N$ as follows:

$$|n^k q^n| = \left(\sqrt[n]{n^k} \cdot |q|\right)^n < s^n \,.$$

However, $\lim\limits_{n\to\infty} s^n = 0$, because (s^n) is a geometric sequence with $0 < s < 1$. Using the majorant principle for zero sequences (see: Theorem 3.26) we obtain that $(n^k q^n)$ is a zero sequence. $\square$

4.21. Corollary. Let $a \in \mathbb{R},\ a > 1$. Applying the previous theorem with $q := \dfrac{1}{a}$ we obtain for any $k \in \mathbb{N}$ that

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0\,.$$

We can express this fact in this way: the exponential function (with base greater than 1) increases faster than a power function of any degree.

In the following theorem we will show that the factorial increases faster than any exponential function.

4.22. Theorem *Let $x \in \mathbb{K}$ be fixed. Then*

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0\,.$$

Proof. Let $n \in \mathbb{N},\ n > |x|$. (There exists such n by the Archimedean property of ordering.) Then we have for any $n \geq N+2$:

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} = \frac{\overbrace{|x| \cdot \ \ldots\ \cdot |x|}^{N \text{ factors}} \cdot |x| \cdot \ \ldots\ \cdot |x|}{\underbrace{1 \cdot \ \ldots\ \cdot N}_{N \text{ factors}} \cdot (N+1) \cdot \ \ldots\ \cdot n} = \frac{|x|^N}{N!} \cdot \frac{|x|}{N+1} \cdot \ \ldots\ \frac{|x|}{n-1} \cdot \frac{|x|}{n} \leq$$

$$\leq \frac{|x|^N}{N!} \cdot \frac{|x|}{n} \longrightarrow 0 \qquad (n \to \infty)\,.$$

In the above estimation we have used that

$$\frac{|x|}{N+1} < 1, \quad \ldots, \quad \frac{|x|}{n-1} < 1\,,$$

and that the other factors of the product are nonnegative.

Finally, applying the majorant principle for zero sequences (see: Theorem 3.26) we obtain that $(\dfrac{x^n}{n!})$ is a zero sequence. $\square$

4.3. Complex Number Sequences

4.23. Definition Let $z_n \in \mathbb{C}$ $(n \in \mathbb{N})$ be a complex number sequence. Let us write its each term in canonical form:

$$z_n = a_n + b_n i \quad (n \in \mathbb{N}),$$

where i denotes the imaginary unit in $\mathbb{C}$: $i = \sqrt{-1}$. Thus we have defined the real number sequences (a_n) and (b_n). Obviously $a_n = \operatorname{Re} z_n$ and $b_n = \operatorname{Im} z_n$.

The sequence (a_n) is called the real part sequence of (z_n). The sequence (b_n) is called the imaginary part sequence of (z_n).

4.24. Example
If $z_n = (n+i)^2$ $(n \in \mathbb{N})$, then

$$z_n = (n+i)^2 = n^2 + 2ni + i^2 = n^2 - 1 + 2ni,$$

thus

$$a_n = \operatorname{Re} z_n = n^2 - 1, \qquad b_n = \operatorname{Im} z_n = 2n \qquad (n \in \mathbb{N}).$$

The following auxiliary theorem will be useful at the discussion of boundedness and convergency of complex sequences.

4.25. Theorem *If $z \in \mathbb{C}$, then*

$$|\operatorname{Re} z| \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z| \quad \textit{and} \quad |\operatorname{Im} z| \le |z| \le |\operatorname{Re} z| + |\operatorname{Im} z|.$$

Proof. Denote by x the real part of z and by y the imaginary part of z respectively. Then

$$|x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z| \quad \text{and} \quad |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} = |z|$$

imply the left-hand inequalities immediately. The common right-hand inequality can be proved as follows:

$$|z| = \sqrt{x^2+y^2} = \sqrt{|x|^2+|y|^2} \le \sqrt{|x|^2+2|x||y|+|y|^2} = \sqrt{(|x|+|y|)^2} =$$
$$= |x| + |y| = |\operatorname{Re} z| + |\operatorname{Im} z|.$$

□

4.26. Theorem *The complex number sequence $z_n \in \mathbb{C}$ $(n \in \mathbb{N})$ is bounded if and only if its real and imaginary part sequences are bounded.*

Proof. Suppose that (z_n) is bounded.Then

$$\exists M > 0\ \forall n \in \mathbb{N} : \quad |z_n| \le M.$$

Using the auxiliary theorem we have

$$|\operatorname{Re} z_n| \le M \quad \text{and} \quad |\operatorname{Im} z_n| \le M \qquad (n \in \mathbb{N}),$$

which means that $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are bounded.

Conversely, suppose that $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are bounded sequences. Then

$$\exists M_1 > 0\ \forall n \in \mathbb{N}: \quad |\operatorname{Re} z_n| \leq M_1, \quad \text{and} \quad \exists M_2 > 0\ \forall n \in \mathbb{N}: \quad |\operatorname{Im} z_n| \leq M_2\,.$$

Once more by the auxiliary theorem we have:

$$|z_n| \leq |\operatorname{Re} z_n| + |\operatorname{Im} z_n| \leq M_1 + M_2\,,$$

which implies the boundedness of (z_n). □

4.27. Theorem *The complex number sequence $z_n = a_n + b_n i \in \mathbb{C}$ $(n \in \mathbb{N})$ is convergent if and only if its real part sequence $(a_n = \operatorname{Re} z_n)$ and its imaginary part sequence $(b_n = \operatorname{Im} z_n)$ both are convergent. In this case:*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \operatorname{Re} z_n + (\lim_{n\to\infty} \operatorname{Im} z_n) \cdot i\,.$$

Proof. Suppose that (z_n) is convergent and denote its limit by $Z = A + Bi \in \mathbb{C}$. Then the sequence

$$z_n - Z = a_n + b_n i - A - Bi = (a_n - A) + (b_n - B)i \qquad (n \in \mathbb{N})$$

is a zero sequence. Using the auxiliary theorem we have

$$|a_n - A| \leq |z_n - Z| \longrightarrow 0 \quad \text{and} \quad |b_n - B| \leq |z_n - Z| \longrightarrow 0 \qquad (n \to \infty)\,.$$

From here it follows – by the majorant principle for zero sequences (see: Theorem 3.26) – that $(a_n - A)$ and $(b_n - B)$ both are zero sequences. This implies that (a_n) and (b_n) are convergent, moreover $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$.

Conversely, suppose that (a_n) and (b_n) are convergent. Then (see: operations with convergent sequences) the linear combination

$$z_n = 1 \cdot a_n + i \cdot b_n \qquad (n \in \mathbb{N})$$

is also convergent and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (a_n + i b_n) = \lim_{n\to\infty} a_n + i \cdot \lim_{n\to\infty} b_n\,.$$

□

4.28. Example Let

$$z_n := \frac{n+1}{n} + \frac{2n+1}{n+1} i \quad (n \in \mathbb{N})\,.$$

Then

$$\operatorname{Re} z_n = \frac{n+1}{n} \longrightarrow 1 \quad (n \to \infty) \quad \text{and} \quad \operatorname{Im} z_n = \frac{2n+1}{n+1} \longrightarrow 2 \quad (n \to \infty)\,,$$

thus by the theorem

$$\lim_{n\to\infty} z_n = 1 + 2i.$$

4.4. Homework

1. Determine the following limits if they exist.

a) $\lim\limits_{n\to\infty} \dfrac{3n^4 - 5n^3 + n^2 - 1}{7(n+1)^4 + n^3 - 7n^2 + 6n - 10}$

b) $\lim\limits_{n\to\infty} \dfrac{(n+2)^6 - (n+3)^6}{(n^2 - 2n - 5)(2n^3 + n^2 + 3)}$

c) $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - n)$

d) $\lim\limits_{n\to\infty} (\sqrt{2n-1} - \sqrt{n+3})$

e) $\lim\limits_{n\to\infty} (\sqrt{2n-1} - \sqrt{2n+3})$

f) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n} - \sqrt{n-1}}$

g) $\lim\limits_{n\to\infty} (\sqrt{n + 4\cdot\sqrt{n}} - \sqrt{n - 10\cdot\sqrt{n}})$

h) $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n+1}{2^n+3}}$

i) $\lim\limits_{n\to\infty} \dfrac{6^{n+2} - 3^{n+1}}{2\cdot 6^n + 5^{n+1}}$

j) $\lim\limits_{n\to\infty} \dfrac{n^3\cdot 2^n + 5^{n+1}}{5^{n-1} - n\cdot 3^n}$

5. Lesson 5

5.1. Monotone Sequences

5.1. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that this sequence is

- monotonically increasing if $\quad \forall n \in \mathbb{N}: \quad a_n \leq a_{n+1}$
- strictly monotonically increasing if $\quad \forall n \in \mathbb{N}: \quad a_n < a_{n+1}$
- monotonically decreasing if $\quad \forall n \in \mathbb{N}: \quad a_n \geq a_{n+1}$
- strictly monotonically decreasing if $\quad \forall n \in \mathbb{N}: \quad a_n > a_{n+1}$
- monotone if it is either monotonically increasing or monotonically decreasing
- strictly monotone if it is either strictly monotonically increasing or strictly monotonically decreasing

5.2. Remarks.

1. The strictly monotonically increasing sequences are often called increasing sequences or strictly increasing sequences.
2. The strictly monotonically decreasing sequences are often called decreasing sequences or strictly decreasing sequences.
3. The monotonically increasing sequences are sometimes called nondecreasing sequences.
4. The monotonically decreasing sequences are sometimes called nonincreasing sequences.

5.3. Theorem *Every real number sequence has a monotone subsequence.*

Proof. Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. Let us call a natural number $k \in \mathbb{N}$ a vertex of (a_n) if

$$\forall n > k : \quad a_k > a_n \,.$$

Case 1: Suppose that the number of vertices is infinite. In this case the vertices form an index sequence:

$$n_1 < n_2 < n_3 < \ldots$$

Since n_1 is a vertex and $n_2 > n_1$, then $a_{n_1} > a_{n_2}$,
since n_2 is a vertex and $n_3 > n_2$, then $a_{n_2} > a_{n_3}$,

and so on. We have obtained a strictly monotonically decreasing subsequence

$$a_{n_1} > a_{n_2} > a_{n_3} > \dots$$

Case 2: Suppose that the number of vertices is finite (may be no vertex). Let n_0 be the last vertex of (a_n) and $n_1 := n_0 + 1$. Then n_1 is no vertex, consequently $\exists\, n_2 > n_1 :$ $a_{n_1} \leq a_{n_2}$,

n_2 is no vertex, consequently $\exists\, n_3 > n_2 :\ a_{n_2} \leq a_{n_3}$,

and so on. We have obtained a monotonically increasing subsequence

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \dots$$

□

5.4. Theorem *Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotone real number sequence. Then it is convergent if and only if it is bounded. Moreover*

$$\lim_{n\to\infty} a_n = \begin{cases} \sup\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically increasing} \\ \\ \inf\{a_n \mid n \in \mathbb{N}\} & \text{if } (a_n) \text{ is monotonically decreasing} \end{cases}$$

Proof. Suppose that (a_n) is convergent. Then it is bounded (see Theorem 3.21).

Conversely, suppose that (a_n) is bounded and let us see the case when (a_n) is monotonically increasing. Because of its boundedness (a_n) is bounded above. Therefore it has finite least upper bound

$$A := \sup\{a_n \mid n \in \mathbb{N}\} \in \mathbb{R}\,.$$

Let $\varepsilon > 0$. Then $A - \varepsilon < A$, thus $A - \varepsilon$ is not upper bound. Therefore

$$\exists\, N \in \mathbb{N} : \quad a_N > A - \varepsilon\,.$$

Since (a_n) is monotonically increasing, then we obtain that $\forall\, n \geq N :\ a_n \geq a_N$. Hence

$$A - \varepsilon < a_N \leq a_n \leq A < A + \varepsilon \qquad (\forall\, n \geq N)\,,$$

which implies $|a_n - A| < \varepsilon$.
This means – by the definition of the limit – that $\lim\limits_{n\to\infty} a_n = A$.

The monotone decreasing case can be proved similarly. □

5.5. Remark. Since every monotone increasing sequence is bounded below, then for a monotone increasing sequence the property „bounded" is equivalent to „bounded above". Similarly, for a monotone decreasing sequence the property „bounded" is equivalent to „bounded below".

5.6. Theorem *[Bolzano-Weierstrass]*

Every bounded number sequence contains a convergent subsequence.

Proof. In the first step we will prove the statement for real number sequences. Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a bounded real number sequence. By Theorem 5.3 it contains a monotone subsequence (a_{n_k}). The subsequence (a_{n_k}) is obviously bounded (with the same bound as (a_n)), therefore it is a monotone and bounded sequence. Consequently – by the previous theorem – (a_{n_k}) is convergent.

In the second step we will prove the statement for complex number sequences. Let $z_n = a_n + b_n i \in \mathbb{C}$ $(n \in \mathbb{N})$ be a bounded complex number sequence. Then by Theorem 4.26 the real sequences (a_n) and (b_n) are bounded. Applying the proved part of the theorem for the real part sequence (a_n), it has a convergent subsequence $(a_{n_k},\ k \in \mathbb{N})$. However, the subsequence $(b_{n_k},\ k \in \mathbb{N})$ of the imaginary part sequence (b_n) is also bounded, so – applying once more the proved part of the theorem – (b_{n_k}) has a convergent subsequence $(b_{n_{k_s}},\ s \in \mathbb{N})$. Hence – using Theorem 4.27 – the complex number sequence

$$z_{n_{k_s}} = a_{n_{k_s}} + b_{n_{k_s}} i \qquad (s \in \mathbb{N})$$

is convergent, and obviously it is a subsequence of (z_n). □

5.2. Euler's Number e

5.7. Theorem *The sequence*

$$a_n := \left(1 + \frac{1}{n}\right)^n \qquad (n \in \mathbb{N})$$

is convergent.

Proof. We want to apply Theorem 5.4, therefore we will prove that (a_n) is increasing and bounded above.

To prove that (a_n) is increasing, let us apply the inequality between the arithmetic and the geometric means (see Theorem 2.8) for the $n+1$ pieces of non-all-equal positive numbers

$$1 + \frac{1}{n},\ 1 + \frac{1}{n},\ \ldots,\ 1 + \frac{1}{n},\ 1\,.$$

We obtain that

$$a_n = \left(1 + \frac{1}{n}\right)^n = \underbrace{\left(1 + \frac{1}{n}\right) \cdot \ldots \cdot \left(1 + \frac{1}{n}\right)}_{n\,\text{times}} \cdot 1 < \left(\frac{n \cdot \left(1 + \frac{1}{n}\right) + 1}{n+1}\right)^{n+1} =$$

$$= \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = a_{n+1}.$$

This means that (a_n) is increasing

To prove that (a_n) is bounded above, let us apply the inequality between the arithmetic and the geometric means for the $n+2$ pieces of non-all-equal positive numbers

$$1+\frac{1}{n},\ 1+\frac{1}{n},\ \dots,\ 1+\frac{1}{n},\ \frac{1}{2},\ \frac{1}{2}\,.$$

We obtain that

$$\frac{1}{4}a_n = \frac{1}{4}\cdot\left(1+\frac{1}{n}\right)^n = \underbrace{\left(1+\frac{1}{n}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right)}_{n\,\text{times}}\cdot\frac{1}{2}\cdot\frac{1}{2} <$$

$$< \left(\frac{n\cdot\left(1+\dfrac{1}{n}\right)+\dfrac{1}{2}+\dfrac{1}{2}}{n+2}\right)^{n+2} = \left(\frac{n+2}{n+2}\right)^{n+2} = 1,$$

Hence we have $a_n < 4\ (n\in\mathbb{N})$. Thus (a_n) is bounded above. $\square$

5.8. Definition On the base of the previous theorem we can define Euler's number e as

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\,.$$

5.9. Remark. Later (see Theorem 9.15) we will prove that the number e is irrational. Its approximating value for three decimal digits is 2.718. This means that

$$|\,e-2.718\,| < \frac{1}{2}\cdot 10^{-3}, \quad \text{that is} \quad 2.7175 < e < 2.7185\,.$$

5.3. Cauchy's Convergence Test

5.10. Definition The number sequence $a_n\in\mathbb{K}\ (n\in\mathbb{N})$ is called a Cauchy sequence if

$$\forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall m,n\geq N:\quad |a_n-a_m|<\varepsilon\,.$$

5.11. Theorem *Every Cauchy sequence is bounded. Consequently – by the Bolzano-Weierstrass theorem – it has convergent subsequence.*

Proof. Let $a_n\in\mathbb{K}\ (n\in\mathbb{N})$ be a Cauchy sequence and let us apply the above definition with $\varepsilon=1$. Then

$$\exists N\in\mathbb{N}\ \forall m,n\geq N:\quad |\,a_n-a_m|<1\,.$$

Let $m:=N$ and apply the second triangle inequality. Thus we have

$$1 > |\,a_n-a_N| \geq \Big|\,|a_n|-|a_N|\,\Big| \geq |a_n|-|a_N| \qquad (n\geq N)\,.$$

Consequently

$$|a_n| < 1+|a_N| \qquad (n\geq N)\,.$$

Hence we have

$$|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|, 1 + |a_N|\} \qquad (n \in \mathbb{N}) .$$

This means that (a_n) is bounded. □

5.12. Theorem *[Cauchy's Convergence Test]*

The number sequence $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ is convergent if and only if it is a Cauchy sequence.

Proof. Suppose that (a_n) is convergent and denote by $A \in \mathbb{K}$ its limit. Furthermore let $\varepsilon > 0$. Then

$$\exists N \in \mathbb{N}\ \forall n \geq N : \quad |a_n - A| < \frac{\varepsilon}{2} .$$

This N will be a good threshold index to prove that (a_n) is a Cauchy sequence. Really, using the first triangle inequality, we have for any $m, n \geq N$:

$$|\,a_n - a_m| = |\,a_n - A + A - a_m| \leq |\,a_n - A| + |\,A - a_m| = |\,a_n - A| + |\,a_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Conversely, suppose that (a_n) is a Cauchy sequence. By the previous theorem (a_n) has a convergent subsequence (a_{n_k}). Denote by A the limit of (a_{n_k}). We will show that $\lim\limits_{n\to\infty} a_n = A$. Let $\varepsilon > 0$.

By the definition of the Cauchy sequence we have

$$\exists N \in \mathbb{N}\ \forall m, n \geq N : \quad |a_n - a_m| < \frac{\varepsilon}{2} .$$

This number N will be a good threshold index to prove that (a_n) tends to A.

Since $\lim\limits_{k\to\infty} a_{n_k} = A$, then

$$\exists K \in \mathbb{N}\ \forall k \geq K : \quad |a_{n_k} - A| < \frac{\varepsilon}{2} .$$

The index sequence (n_k) is unbounded above, therefore

$$\exists\, k \in \mathbb{N} : \quad k \geq K \quad \text{and} \quad n_k \geq N.$$

Fix such a k. Then we have for any $n \geq N$ – since in this case n, $n_k \geq N$ – the following inequalities:

$$|\,a_n - a_{n_k}| < \frac{\varepsilon}{2} \quad \text{and} \quad |\,a_{n_k} - A| < \frac{\varepsilon}{2} .$$

Finally, using the first triangle inequality we have

$$|\,a_n - A| = |\,a_n - a_{n_k} + a_{n_k} - A| \leq |\,a_n - a_{n_k}| + |\,a_{n_k} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This means that $\lim\limits_{n\to\infty} a_n = A$. □

5.13. Remark. The fact that every Cauchy sequence is convergent is a typical property of the real numbers. This property is known as $\mathbb{R}$ is a complete metric space.

5.4. Infinite Limits of Real Number Sequences

5.14. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that (a_n) tends to $+\infty$ if

$$\forall P > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : \quad a_n > P \, .$$

This fact is denoted in one of the following ways:

$$\lim a = +\infty \, , \quad \lim a_n = +\infty \, , \quad \lim_{n\to\infty} a_n = +\infty \, , \quad a_n \to +\infty \quad (n \to \infty) \, ,$$

$$\lim(a_n) = +\infty \, , \quad (a_n) \to +\infty \quad (n \to \infty) \, .$$

5.15. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that (a_n) tends to $-\infty$ if

$$\forall P < 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : \quad a_n < P \, .$$

This fact is denoted in one of the following ways:

$$\lim a = -\infty \, , \quad \lim a_n = -\infty \, , \quad \lim_{n\to\infty} a_n = -\infty \, , \quad a_n \to -\infty \quad (n \to \infty) \, ,$$

$$\lim(a_n) = -\infty \, , \quad (a_n) \to -\infty \quad (n \to \infty) \, .$$

5.16. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real number sequence. We say that (a_n) has a limit if

$$(a_n) \text{ is convergent} \quad \text{or} \quad \lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty \, .$$

In other words:

$$\exists \, A \in \overline{\mathbb{R}} : \quad \lim_{n\to\infty} a_n = A \, .$$

5.17. Remark. Define the neighbourhoods of $+\infty$ and of $-\infty$ with radius $r > 0$ as follows:

$$B(+\infty, r) := \{x \in \mathbb{R} \mid \; x > \frac{1}{r} \, \} = (\frac{1}{r}, \, +\infty) \subset \mathbb{R} \, ,$$

and

$$B(-\infty, r) := \{x \in \mathbb{R} \mid \; x < -\frac{1}{r} \, \} = (-\infty, \, -\frac{1}{r}) \subset \mathbb{R} \, .$$

Using these definitions we can express uniformly that $\lim\limits_{n\to\infty} a_n = A$.

Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a real sequence and $A \in \overline{\mathbb{R}}$. Then $\lim\limits_{n\to\infty} a_n = A$ is equivalent to

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : \quad a_n \in B(A, \, \varepsilon) \, .$$

5.18. Remark. It is obvious that if a sequence tends to $+\infty$, then it is unbounded above. If a sequence tends to $-\infty$, then it is unbounded below. The converse statement is not true as we can see from the example $a_n := (-1)^n \cdot n, \; (n \in \mathbb{N})$.

5.19. Theorem *a) Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotonically increasing sequence. Suppose that it is not bounded above. Then*

$$\lim_{n\to\infty} a_n = +\infty\,.$$

b) Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be a monotonically decreasing sequence. Suppose that it is not bounded below. Then

$$\lim_{n\to\infty} a_n = -\infty\,.$$

Proof. We will prove part a). Let $P > 0$. Since (a_n) is not bounded above, then

$$\exists\, N \in \mathbb{N}: \quad a_N > P.$$

However, (a_n) is monotone increasing, therefore

$$\forall\, n \ge N: \quad a_n \ge a_N > P\,.$$

This means that $\lim\limits_{n\to\infty} a_n = +\infty$.

Part b) can be proved similarly. □

5.20. Corollary. Combining this theorem with Theorem 5.4 we obtain that every monotone sequence has a limit. If the sequence is bounded, then this limit is finite, if it is unbounded, then the limit is infinite.

Now we will discuss the algebraic operations with infinite limits.

5.21. Theorem *[Addition]*

Let $a_n, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Suppose that $\lim\limits_{n\to\infty} a_n = A$ where $-\infty < A \le +\infty$ and $\lim\limits_{n\to\infty} b_n = +\infty$. Then

$$\lim_{n\to\infty} (a_n + b_n) = +\infty\,.$$

Proof. Let us fix a real number $P_1 < A$. Then by definition of the limit

$$\exists\, N_1 \in \mathbb{N}\ \forall\, n \ge N_1: \quad a_n > P_1\,.$$

Let $P > 0$. Since $\lim\limits_{n\to\infty} b_n = +\infty$, then to the real number $P - P_1$

$$\exists\, N_2 \in \mathbb{N}\ \forall\, n \ge N_2: \quad b_n > P - P_1\,.$$

Let $N := \max\{N_1, N_2\}$. Then for every $n \ge N$ holds

$$a_n + b_n > P_1 + P - P_1 = P\,.$$

This means that $\lim\limits_{n\to\infty} (a_n + b_n) = +\infty$. □

5.22. Remarks.

1. The theorem does not state anything about the case when $A = -\infty$. It is not by chance, in this case the sequence $(a_n + b_n)$ can behave itself variously, as the following table shows:

a_n	$A = \lim a_n$	b_n	$B = \lim b_n$	$a_n + b_n$	$\lim(a_n + b_n)$
$-n + c$ ($c \in \mathbb{R}$ arbitrarily fixed)	$-\infty$	n	$+\infty$	c	c
$-2n$	$-\infty$	n	$+\infty$	$-n$	$-\infty$
$-n$	$-\infty$	$2n$	$+\infty$	n	$+\infty$
$-n$	$-\infty$	$n + (-1)^n$	$+\infty$	$(-1)^n$	does not exist (and bounded)
$-n^2$	$-\infty$	$n^2 + (-1)^n \cdot n$	$+\infty$	$(-1)^n \cdot n$	does not exist (and unbounded)

2. Our theorem makes it possible for us to extend the addition from $\mathbb{R}$ into $\overline{\mathbb{R}}$ preserving the identity

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \,.$$

The table of addition in $\overline{\mathbb{R}}$ is the following:

$x+y$	$y=-\infty$	$y \in \mathbb{R}$	$y=+\infty$
$x=-\infty$	$-\infty$	$-\infty$	not defined
$x \in \mathbb{R}$	$-\infty$	$x+y$	$+\infty$
$x=+\infty$	not defined	$+\infty$	$+\infty$

You can see that the sum of the elements $-\infty$ and $+\infty$ is undefined. In all other cases the sum is defined.

3. After these definitions the theorem about the limit of the sum of two real sequences having limits can be stated shortly as

5.23. Theorem *If the real number sequences (a_n) and (b_n) have limits, furthermore the sum of $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exists, then*

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\,.$$

4. The sums $(-\infty)+(+\infty)$ and $(+\infty)+(-\infty)$ are called indeterminate sums. They are a kind of indeterminate expressions.

5. Using the connection between the addition and subtraction we can reduce the subtraction of sequences having infinite limits to addition. Accordingly the differences $(+\infty)-(+\infty)$ and $(-\infty)-(-\infty)$ are also indeterminate expressions.

5.24. Theorem *[Multiplication]*
Let $a_n,\, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Suppose that $\lim\limits_{n\to\infty} a_n = A$ where $A \in \overline{\mathbb{R}}\setminus\{0\}$ and $\lim\limits_{n\to\infty} b_n = +\infty$. Then

$$\lim_{n\to\infty}(a_nb_n) = \begin{cases} +\infty & \textit{if} \quad A>0 \\ -\infty & \textit{if} \quad A<0 \end{cases}$$

Proof. We will prove the case $0 < A \le +\infty$. The proof of the other case is similar.

Let us fix a real number P_1 such that $0 < P_1 < A$. Then by definition of the limit

$$\exists\, N_1 \in \mathbb{N}\ \forall\, n \ge N_1 : \quad a_n > P_1\,.$$

Let $P > 0$. Since $\lim\limits_{n\to\infty} b_n = +\infty$, then to the real number $\dfrac{P}{P_1}$

$$\exists\, N_2 \in \mathbb{N}\ \forall\, n \geq N_2 : \quad b_n > \frac{P}{P_1}\,.$$

Let $N := \max\{N_1, N_2\}$. Then for every $n \geq N$ holds

$$a_n b_n > a_n \cdot \frac{P}{P_1} > P_1 \cdot \frac{P}{P_1} = P\,.$$

This means that $\lim\limits_{n\to\infty} (a_n b_n) = +\infty$. □

5.25. Remark. From our theorem we can deduce that if we define the multiplication in $\overline{\mathbb{R}}$ as follows

$x \cdot y$	$y = -\infty$	$-\infty < y < 0$	$y = 0$	$0 < y < +\infty$	$y = +\infty$
$x = -\infty$	$+\infty$	$+\infty$	not defined	$-\infty$	$-\infty$
$-\infty < x < 0$	$+\infty$	xy	0	xy	$-\infty$
$x = 0$	not defined	0	0	0	not defined
$0 < x < +\infty$	$-\infty$	xy	0	xy	$+\infty$
$x = +\infty$	$-\infty$	$-\infty$	not defined	$+\infty$	$+\infty$

then we have the following theorem

5.26. Theorem *If the real number sequences (a_n) and (b_n) have limits, furthermore the product of $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exists, then*

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right).$$

Accordingly the products $0\cdot(+\infty)$, $0\cdot(-\infty)$, $(+\infty)\cdot 0$ and $(-\infty)\cdot 0$ are indeterminate expressions.

5.27. Theorem *[Reciprocal] Let $b_n \in \mathbb{R} \setminus \{0\}$, $(n \in \mathbb{N})$ be a real number sequence. Suppose that it has a limit and $\lim\limits_{n\to\infty} b_n = B \in \overline{\mathbb{R}}$. Then*

$$\lim_{n\to\infty}\left(\frac{1}{b_n}\right) = \begin{cases} \dfrac{1}{B} & \textit{if} \quad B \notin \{-\infty\,,\,0\,,\,+\infty\} \\[2ex] 0 & \textit{if} \quad B \in \{-\infty\,,\,+\infty\} \\[2ex] +\infty & \textit{if} \quad B = 0 \textit{ and } \exists\, N_0 \in \mathbb{N}\; \forall\, n \geq N_0: \quad b_n > 0 \\[2ex] -\infty & \textit{if} \quad B = 0 \textit{ and } \exists\, N_0 \in \mathbb{N}\; \forall\, n \geq N_0: \quad b_n < 0 \\[2ex] \textit{does not exist} & \textit{if} \quad B = 0 \textit{ and there are infinitely many positive} \\ & \qquad\qquad \textit{and infinitely many negative terms in the sequence} \end{cases}$$

Proof. The case $B \notin \{-\infty\,,\,0\,,\,+\infty\}$ was proved at the convergence sequences (see Theorem 4.8).

Suppose that $B = +\infty$. Let $\varepsilon > 0$. Then

$$\exists N \in \mathbb{N}\; \forall n \geq N: \quad b_n > \frac{1}{\varepsilon}\,.$$

After rearranging we have $-\varepsilon < 0 < \dfrac{1}{b_n} < \varepsilon$, that is $\left|\dfrac{1}{b_n} - 0\right| < \varepsilon$.

The case $B = -\infty$ can be proved similarly.

Suppose that

$$B = 0 \quad \text{and} \quad \exists\, N_0 \in \mathbb{N}\; \forall\, n \geq N_0: \quad b_n > 0\,.$$

Let $P > 0$. Then

$$\exists N_1 \in \mathbb{N}\; \forall n \geq N_1: \quad b_n < \frac{1}{P}\,,$$

thus for any $n \geq N := \max\{N_0,\, N_1\}$ holds $0 < b_n < \dfrac{1}{P}$. After rearranging we have $\dfrac{1}{b_n} > P$ $(n \geq N)$.

The case when $B = 0$ and $\exists\, N_0 \in \mathbb{N}\; \forall\, n \geq N_0: \quad b_n < 0$ can be proved similarly.

Finally, suppose that the sequence (b_n) contains infinitely many positive and infinitely many negative terms and $\lim\limits_{n\to\infty} b_n = 0$. In this case the subsequence of the reciprocals of the positive terms tends to $+\infty$, the subsequence of the reciprocals of the negative terms tends to $-\infty$. Thus $(\dfrac{1}{b_n})$ has no limit. $\square$

5.28. Example

If $q \in \mathbb{R}$, $q > 1$, then the geometric sequence (q^n) tends to $+\infty$. Really, in this case

$$q^n = \frac{1}{(\frac{1}{q})^n} \quad \text{and} \quad 0 < \frac{1}{q} < 1$$

Thus $\lim\limits_{n\to\infty} \left(\frac{1}{q}\right)^n = 0$ and $\left(\frac{1}{q}\right)^n > 0$ $(n \in \mathbb{N})$. Using the previous theorem we obtain

$$\lim_{n\to\infty} q^n = +\infty\,.$$

The quotients having infinite limits can be discussed by combining the results about the reciprocal and the product.

Finally let us collect the undefined (indeterminate) expressions:

Sums: $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$

Differences: $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$

Products: $0 \cdot (\pm\infty)$, $(\pm\infty) \cdot 0$

Quotients: $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$

If a limit problem results in an indeterminate form, then we need to transform the formula of the sequence into a non-indeterminate form.

5.5. Homework

1. Determine the limits of the following recursive sequences if they exist.

$$a)\quad a_1 = 0, \quad a_{n+1} = \sqrt{4 + 3a_n} \qquad b)\quad a_1 = 1, \quad a_{n+1} = 1 + \frac{a_n^2}{4}$$

2. Determine the following limits if they exist.

$$a)\quad \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^{2n+5} \qquad b)\quad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n+3}$$

$$c)\quad \lim_{n\to\infty} \left(\frac{3n-4}{3n+5}\right)^{4n+2} \qquad d)\quad \lim_{n\to\infty} \left(\frac{2n+1}{2n-3}\right)^{3n-2}$$

3. Prove by definition of the limit that

$$a)\quad \lim_{n\to\infty} \frac{n^3 - 3n^2 + n - 1}{5n^2 + n - 3} = +\infty \qquad b)\quad \lim_{n\to\infty} \frac{n^4 - 4n^3 + 3n + 2}{1 - 7n - 4n^2} = -\infty$$

In question a) determine a threshold index to $P = 1000$. In question b) determine a threshold index to $P = -2000$.

6. Lesson 6

6.1. Numerical Series

In this chapter we will discuss the problem that the terms of a numerical sequence can be added in some sense.

6.1. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ be a number sequence. The expression

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called an infinite numerical sum or an infinite numerical series. The numbers a_n are the terms of the series. The sequence

$$S_n := a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k \qquad (n \in \mathbb{N})$$

is called the partial sum sequence of the series. S_n is the n-th partial sum.

6.2. Remark. As mentioned at the sequences, the starting index is not necessarily 1. If the starting index is the integer $p \in \mathbb{Z}$, then the series is

$$a_p + a_{p+1} + a_{p+2} \ldots = \sum_{n=p}^{\infty} a_n \,,$$

and the partial sum sequence is

$$S_n := a_p + a_{p+1} + \ldots + a_n = \sum_{k=p}^{n} a_k \qquad (n \in \mathbb{Z},\ n \geq p)\,.$$

If $p \neq 1$, then it is better to say for S_n „the partial sum according to the index n" instead of „the n-th partial sum".

6.3. Definition The series $\sum\limits_{n=1}^{\infty} a_n$ is called convergent if its partial sum sequence (S_n) is convergent. In this case the limit of the partial sum sequence is called the sum of the series. For the sum of the series we will use the same symbol as for the series itself:

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \,.$$

A series is called divergent if it is not convergent. In this case the sum of the infinitely many terms a_n is undefined.

6.4. Theorem *Let a_n, $b_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and suppose that the series* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=1}^{\infty} b_n$ *are convergent. Then*

a) The series $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ *is also convergent and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \,.$$

b) For any $c \in \mathbb{K}$ the series $\sum\limits_{n=1}^{\infty} (c \cdot a_n)$ *is also convergent and*

$$\sum_{n=1}^{\infty}(c \cdot a_n) = c \cdot \sum_{n=1}^{\infty} a_n \,.$$

Proof.

a) We have for any $n \in \mathbb{N}$ that

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \,.$$

Therefore (taking $n \to \infty$)

$$\begin{aligned}\sum_{n=1}^{\infty}(a_n + b_n) &= \lim_{n\to\infty} \sum_{k=1}^{n}(a_k + b_k) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \right) = \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \,.\end{aligned}$$

b) Using a similar idea we have for any $n \in \mathbb{N}$ that

$$\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \sum_{k=1}^{n} a_k \,.$$

Therefore (taking $n \to \infty$)

$$\sum_{n=1}^{\infty}(c \cdot a_n) = \lim_{n\to\infty} \sum_{k=1}^{n}(c \cdot a_k) = \lim_{n\to\infty} \left(c \cdot \sum_{k=1}^{n} a_k \right) = c \cdot \lim_{n\to\infty} \sum_{k=1}^{n} a_k = c \cdot \sum_{n=1}^{\infty} a_n \,.$$

$\square$

6.2. Geometric Series

An important type of convergent series is the geometric series. This is the series whose terms form a geometric sequence.

6.5. Definition Let $q \in \mathbb{K}$ be a fixed number. Then the series

$$q + q^2 + q^3 + \ldots = \sum_{n=1}^{\infty} q^n$$

is called a geometric series (with base q or with quotient q).

6.6. Theorem *The geometric series is convergent if and only if* $|q| < 1$. *In this case*

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}.$$

Proof. Suppose that $q = 1$. Then

$$S_n = \sum_{k=1}^{n} 1^k = n \qquad (n \in \mathbb{N}),$$

which is an obviously divergent sequence.

Suppose that $q \neq 1$. Using the formula for the sum of the first n terms of a geometric sequence (the formula was proved in secondary school), we have

$$S_n = \sum_{k=1}^{n} q^k = q \cdot \frac{q^n - 1}{q - 1} \qquad (n \in \mathbb{N}).$$

It follows from the theorem about the convergency of a geometric sequence (see Theorem 4.16) that if $|q| \geq 1$, then (S_n) is divergent. Furthermore if $|q| < 1$, then we have

$$\sum_{n=1}^{\infty} q^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} q \cdot \frac{q^n - 1}{q - 1} = q \cdot \frac{0 - 1}{q - 1} = \frac{q}{1-q}.$$

□

6.7. Remark. In many cases the indices of the geometric series start with 0. Then – using a proof similar to the previous one – we have the following

6.8. Theorem *The geometric series*

$$1 + q + q^2 + q^3 + \ldots = \sum_{n=0}^{\infty} q^n$$

is convergent if and only if $|q| < 1$. *In this case*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

Remark that this formula is valid for $q = 0$ too, if we agree that the first term q^0 of the series denotes the number 1 for any $q \in \mathbb{K}$, independently of the fact that the power 0^0 is undefined.

6.3. The Zero Sequence Test and the Cauchy Criterion

6.9. Theorem *[Zero Sequence Test]*

Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and suppose that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0\,.$$

Proof. Let $A := \sum\limits_{n=1}^{\infty} a_n$ and (S_n) be the partial sum sequence. Obviously

$$S_n = S_{n-1} + a_n \quad (n \in \mathbb{N},\ n \geq 2)\,.$$

Moreover we have

$$\lim_{n\to\infty} S_n = A \quad \text{and} \quad \lim_{n\to\infty} S_{n-1} = A\,,$$

therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = A - A = 0\,.$$

□

6.10. Remark. The converse statement is not true. As a counterexample let us study the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}\,.$$

The terms of this series obviously tend to 0. We will show that the harmonic series is divergent.

Denote by S_n its partial sums:

$$S_n := \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \,\dots\, + \frac{1}{n} \qquad (n \in \mathbb{N})\,.$$

We will prove that the sequence (S_n) is not bounded above. To see this, it is enough to show that the subsequence $(S_{2^n},\, n \in \mathbb{N})$ is not bounded above. Let us see the following estimation:

$$\begin{aligned}
S_{2^n} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \,\dots\, + \frac{1}{2^n} = \\
&= \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \,\dots\, + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + \,\dots\, + \frac{1}{2^n}\right) \geq \\
&\geq \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \,\dots\, + \left(\frac{1}{2^n} + \frac{1}{2^n} + \,\dots\, + \frac{1}{2^n}\right) = \\
&= 1 + 2^0 \cdot \frac{1}{2} + 2^1 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{8} + \,\dots\, + 2^{n-1} \cdot \frac{1}{2^n} = 1 + n \cdot \frac{1}{2} > \frac{n}{2} \qquad (n \in \mathbb{N})\,.
\end{aligned}$$

The sequence $(\frac{n}{2})$ is obviously not bounded above, consequently so does (S_{2^n}).

6.11. Theorem *[Cauchy Criterion]*

Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. Then the series $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m \geq n \geq N : \quad |a_n + a_{n+1} + \ldots + a_m| < \varepsilon\,. \tag{6.1}$$

The number N is called threshold index.

Proof. Denote by (S_n) the sequence of the partial sums: $S_n = a_1 + a_2 + \ldots + a_n$. Since for $m \geq n \geq 2$

$$a_n + a_{n+1} + \ldots + a_m = S_m - S_{n-1}\,,$$

then the statement is a simple consequence of the application of Cauchy's convergence test (see Theorem 5.12) for the sequence (S_n). □

The following theorem is a simple corollary of the Cauchy criterion.

6.12. Theorem *Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$.*

Let $n_k \in \mathbb{N} \quad (k \in \mathbb{N})$ and $m_k \in \mathbb{N} \quad (k \in \mathbb{N})$ be two index sequences (see Definition 3.3). Suppose that

$$n_k \leq m_k \qquad (k \in \mathbb{N})\,.$$

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then*

$$\lim_{k\to\infty} \sum_{i=n_k}^{m_k} a_i = 0\,.$$

Proof. Let $\varepsilon > 0$. Since the series is convergent, then by (6.1)

$$\exists N \in \mathbb{N}\ \forall m \geq n \geq N : \quad \left| \sum_{i=n}^{m} a_i \right| < \varepsilon\,.$$

However, $\lim_{k\to\infty} n_k = +\infty$, therefore

$$\exists K \in \mathbb{N}\ \forall k \geq K : \quad m_k \geq n_k \geq N\,.$$

Thus we have proved that

$$\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall k \geq K : \quad \left| \sum_{i=n_k}^{m_k} a_i \right| < \varepsilon\,,$$

which implies the statement of the theorem. □

6.13. Corollary. If we apply the above result for the index sequences $n_k = m_k = k \quad (k \in \mathbb{N})$, then we obtain once more the zero sequence test.

6.4. Positive Term Series

6.14. Definition Let $a_n \in \mathbb{R}$ $(n \in \mathbb{N})$. The series $\sum\limits_{n=1}^{\infty} a_n$ is called a positive term series if $a_n \geq 0$ $(n \in \mathbb{N})$.

6.15. Theorem *A positive term series is convergent if and only if its partial sum sequence is bounded above.*

Proof. Denote by $\sum\limits_{n=1}^{\infty} a_n$ a positive term series and by (S_n) its partial sum sequence. Since by $a_{n+1} \geq 0$

$$S_{n+1} = S_n + a_{n+1} \geq S_n \,,$$

then (S_n) is monotone increasing. Thus by Theorem 5.4 it is convergent if and only if it is bounded above. $\square$

6.16. Remark. By the previous theorem the only possibility for divergency of a positive term series is that $\lim\limits_{n\to\infty} S_n = +\infty$. By this reason we use the following notations for positive term series:

- $\sum\limits_{n=1}^{\infty} a_n < \infty$ if $\sum\limits_{n=1}^{\infty} a_n$ is convergent
- $\sum\limits_{n=1}^{\infty} a_n = \infty$ if $\sum\limits_{n=1}^{\infty} a_n$ is divergent

6.17. Theorem *[Direct Comparison Tests]*
Let $a_n, b_n \in \mathbb{R}$ $(n \in \mathbb{N})$ and suppose that $0 \leq a_n \leq b_n$ $(n \in \mathbb{N})$. Then

a) If $\sum\limits_{n=1}^{\infty} b_n < \infty$, then $\sum\limits_{n=1}^{\infty} a_n < \infty$ (Majorant Criterion)

b) If $\sum\limits_{n=1}^{\infty} a_n = \infty$, then $\sum\limits_{n=1}^{\infty} b_n = \infty$ (Minorant Criterion)

Proof. Denote by S_n and by T_n the partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ respectively. By the assumptions of the theorem (S_n) and (T_n) are monotone increasing, furthermore

$$S_n \leq T_n \qquad (n \in \mathbb{N})\,. \tag{6.2}$$

a) If $\sum\limits_{n=1}^{\infty} b_n < \infty$, then (T_n) is bounded above. Therefore by (6.2) (S_n) is also bounded above. Consequently $\sum\limits_{n=1}^{\infty} a_n < \infty$.

b) If $\sum\limits_{n=1}^{\infty} a_n = \infty$, then (S_n) is not bounded above. Therefore by (6.2) (T_n) is also not bounded above. Consequently $\sum\limits_{n=1}^{\infty} b_n = \infty$.

□

6.18. Remarks.

1. The statement of the theorem will be obviously true if the condition $0 \le a_n \le b_n$ holds except for a finite number of indices n.
2. If a real number series does not contain infinitely many positive and infinitely many negative terms, then it can be investigated as a positive term series. Namely, if every term is negative, then we factor out (-1) from the series. If the series contains a finite number of positive or a finite number of negative terms, then we leave these terms.

6.5. The Hyperharmonic Series

6.19. Definition Let $p > 0$ be a fixed real number. The positive term series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a hyperharmonic series or a p-series.

To investigate the convergence of the hyperharmonic series, first we prove an interesting auxiliary theorem.

6.20. Theorem *[Cauchy's Condensation Principle]*
Let $a_n \ge 0$ $(n \in \mathbb{N})$ be a monotone decreasing sequence of nonnegative numbers. Then

$$\sum_{n=1}^{\infty} a_n < \infty \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} 2^k \cdot a_{2^k} < \infty\,.$$

Proof. Both series are positive term series, therefore we will apply Theorem 6.15.
Let us denote by S_n and by T_k the partial sums:

$$S_n := a_1 + \ldots + a_n \quad \text{and} \quad T_k := 2^0 \cdot a_{2^0} + 2^1 \cdot a_{2^1} + \ldots + 2^k \cdot a_{2^k}\,.$$

First suppose that $\sum\limits_{k=0}^{\infty} 2^k \cdot a_{2^k} < \infty$. Then (T_k) is bounded above. Denote by M one of its upper bounds. Then for any $k \in \mathbb{N}$ holds

$$\begin{aligned}
&S_{2^k-1} = a_1 + \ldots + a_{2^k-1} = \\
&= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + (a_{2^{k-1}} + a_{2^{k-1}+1} + \ldots + a_{2^k-1}) \le \\
&\le a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \ldots + (a_{2^{k-1}} + a_{2^{k-1}} + \ldots + a_{2^{k-1}}) = \\
&= 1a_1 + 2a_2 + 4a_4 + \ldots + 2^{k-1} \cdot a_{2^{k-1}} = T_{k-1} \le M\,.
\end{aligned}$$

Therefore the subsequence (S_{2^k-1}) of (S_n) is bounded above. However, (S_n) is monotone increasing, which implies that (S_n) is bounded above, consequently $\sum\limits_{n=1}^{\infty} a_n < \infty$.

Conversely, suppose that $\sum\limits_{n=1}^{\infty} a_n < \infty$. Then (S_n) is bounded above. Denote by M one of its upper bounds. Then for any $k \in \mathbb{N}$ holds

$$\begin{aligned}
&S_{2^k} = a_1 + \ldots + a_{2^k} = \\
&= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k}) \geq \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \ldots + (a_{2^k} + a_{2^k} + \ldots + a_{2^k}) = \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{k-1} \cdot a_{2^k} = \\
&= a_1 + \frac{2a_2 + 4a_4 + 8a_8 + \ldots + 2^k \cdot a_{2^k}}{2} = a_1 + \frac{T_k - a_1}{2} = \frac{a_1 + T_k}{2} \geq \frac{T_k}{2}\,.
\end{aligned}$$

After rearranging we obtain that

$$T_k \leq 2S_{2^k} \leq 2M\,.$$

Therefore (T_k) is bounded above, consequently $\sum\limits_{k=0}^{\infty} 2^k \cdot a_{2^k} < \infty$. □

Using this auxiliary theorem we can prove the convergence theorem of hyperharmonic series.

6.21. Theorem *Let $p > 0$. The hyperharmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if and only if $p > 1$.

Proof. We can apply Cauchy's condensation principle, consequently

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} < \infty\,.$$

However, the series on the right side can be transformed as

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{2^k}{(2^p)^k} = \sum_{k=0}^{\infty} \left(\frac{2}{2^p}\right)^k = \sum_{k=0}^{\infty} (2^{1-p})^k\,.$$

This geometric series is convergent if and only if $2^{1-p} < 1$, that is if $p > 1$. □

6.22. Remark. Using Theorem 6.12, we can present a simple proof for divergence of the hyperharmonic series in the case $0 < p \leq 1$.

Let $0 < p \leq 1$. Then $1 - p \geq 0$, and we have for any $n \in \mathbb{N}$:

$$\begin{aligned}
\sum_{i=n+1}^{2n} \frac{1}{i^p} &= \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + \ldots + \frac{1}{(n+n)^p} \geq \\
&\geq \underbrace{\frac{1}{(n+n)^p} + \frac{1}{(n+n)^p} + \ldots + \frac{1}{(n+n)^p}}_{n\,\text{times}} = n \cdot \frac{1}{(2n)^p} = \frac{n^{1-p}}{2^p} \geq \frac{1}{2^p}\,.
\end{aligned}$$

Thus the sequence

$$\sum_{i=n+1}^{2n} \frac{1}{i^p} \qquad (n \in \mathbb{N})$$

is not a zero sequence. Consequently – using Theorem 6.12 – the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is divergent.

6.6. Alternating Series

A real number series is called an alternating series if it contains alternately positive and negative terms.

6.23. Definition Let $a_1 \geq a_2 \geq a_3 \geq \ldots > 0$ be a monotonically decreasing sequence of positive numbers. Then the series

$$a_1 - a_2 + a_3 - a_4 + a_5 - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$$

is called a series of Leibniz type.

6.24. Theorem *[Leibniz Criterion]*
The series of Leibniz type

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$$

is convergent if and only if $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Suppose that $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$ is convergent. Then by Theorems 6.9 and 3.25 $\lim\limits_{n\to\infty} a_n = 0$ holds.

Conversely, suppose that $\lim\limits_{n\to\infty} a_n = 0$.

Denote by S_n the partial sums

$$S_n = \sum_{k=1}^{\infty} (-1)^{k-1} \cdot a_k \qquad (n \in \mathbb{N}),$$

and let us discuss the subsequences (S_{2n}) and (S_{2n-1}).

Since

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n} \quad \text{and}$$
$$S_{2n+1} = S_{2n-1} - (a_{2n} - a_{2n+1}) \leq S_{2n-1} \quad (n \in \mathbb{N}),$$

then the subsequence (S_{2n}) is monotone increasing and the subsequence (S_{2n-1}) is monotone decreasing. Moreover by

$$S_2 \leq S_{2n} \leq S_{2n} + a_{2n+1} = S_{2n+1} \leq S_1 \quad (n \in \mathbb{N})$$

the sequence (S_{2n}) is bounded above and the sequence (S_{2n+1}) is bounded below. Thus both are convergent. Let

$$A := \lim_{n\to\infty} S_{2n} \quad \text{and} \quad B := \lim_{n\to\infty} S_{2n-1}\,.$$

Taking $n \to \infty$ in the inequality

$$S_{2n+1} = S_{2n} + a_{2n+1}$$

(using $\lim\limits_{n\to\infty} a_{2n+1} = 0$) we have $A = B$.

Since (S_{2n}) and (S_{2n+1}) are complement subsequences, then $\lim\limits_{n\to\infty} a_n = A = B$. $\square$

6.25. Example

If we add the terms of the hyperharmonic series with alternating signs, we obtain a convergent series of Leibniz type. More precisely, if $p > 0$ is fixed, then the alternating hyperharmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = \frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

is convergent. Especially (for $p = 1$) the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is convergent.

6.7. Homework

1. Determine the partial sums and the sums of the following series.

$a)$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{n+1} - 4^{n+2}}{6^{n+3}}$ $\qquad$ $b)$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1} - 3 \cdot 2^{n+3}}{6 \cdot 5^n}$

$c)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+3)}$ $\qquad$ $d)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-2) \cdot (3n+1)}$

2. Determine whether the following series are convergent or not.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+3n^2-4n+7}{10\sqrt{n^9}+n^3-5n^2-6n+4}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+6n+2}{3\sqrt{n^6+3}-4n^2-n+5}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{(-2)^n\cdot(n^2-n+1)}$

f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n\cdot\left(\frac{n}{n+1}\right)^n$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{\binom{n}{2}}{\binom{n}{4}}$

7. Lesson 7

7.1. Absolute and Conditional Convergence

At the end of the previous section (see: Example 6.25) we have seen that the alternating harmonic series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent. However, (see Remark 6.10) the series consisting of the absolute values of its terms

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. Thus the convergence of the series consisting of the absolute values seems to be a stronger requirement than the common convergence of a series.

7.1. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. The series $\sum\limits_{n=1}^{\infty} a_n$ is called absolutely convergent, if the series of absolute values

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

7.2. Theorem *Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$. If the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

Proof. Since $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, then by the Cauchy criterion (see Theorem 6.11) we have

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}\ \forall\, m \geq n \geq N: \quad |a_n|+|a_{n+1}|+\ldots+|a_m| = \Big|\, |a_n|+|a_{n+1}|+\ldots+|a_m| \,\Big| < \varepsilon\,.$$

Using the first triangle inequality,

$$|a_n + a_{n+1} + \ \ldots\ + a_m| \leq |a_n| + |a_{n+1}| + \ \ldots\ + |a_m|\,,$$

thus we obtain that

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}\ \forall\, m \geq n \geq N: \quad |a_n + a_{n+1} + \ \ldots\ + a_m| < \varepsilon\,.$$

Consequently – once more using the Cauchy criterion – the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

$\square$

7.3. Remark. The converse statement is not true as we have seen at the beginning of the section via the counterexample $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$.

7.4. Definition A numerical series is called conditionally convergent if it is convergent, but not absolutely convergent.

7.5. Example

The alternating hyperharmonic series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$ is absolutely convergent if $p > 1$, and it is conditionally convergent if $0 < p \leq 1$.

7.6. Remark. If a real number series does not contain infinitely many positive and infinitely many negative terms, then it is convergent if and only if it is absolutely convergent. In this case the convergence and the absolute convergence are equivalent.

It is natural that a finite sum can be rearranged arbitrarily without changing the result of the addition. Now we present two theorems about the rearrangement of the infinite sums.

7.7. Theorem *Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$, and $p : \mathbb{N} \to \mathbb{N}$ be a bijection. If the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then the series*

$$\sum_{i=1}^{\infty} a_{p(i)}$$

is absolutely convergent.

The series $\sum\limits_{i=1}^{\infty} a_{p(i)}$ is called a rearrangement of $\sum\limits_{n=1}^{\infty} a_n$.

Proof. Since the absolutely convergence the quantity $K := \sum\limits_{n=1}^{\infty} |a_n|$ is finite. Furthermore, for any $n \in \mathbb{N}$ let

$$M = M(n) := \max\{p(1),\, p(2),\, \dots,\, p(n)\}\,.$$

Then

$$\{p(1),\, p(2),\, \dots,\, p(n)\} \subseteq \{1,\, 2,\, \dots,\, M\}\,,$$

therefore

$$\sum_{i=1}^{n} |a_{p(i)}| \leq \sum_{k=1}^{M} |a_k| \leq K \qquad (n \in \mathbb{N})\,.$$

Thus the partial sum sequence $\left(\sum\limits_{i=1}^{n} |a_{p(i)}|\right)$ is bounded above, the proof is complete. $\square$

7.8. Theorem *Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$, and $p : \mathbb{N} \to \mathbb{N}$ be a bijection. If the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then*

$$\sum_{i=1}^{\infty} a_{p(i)} = \sum_{n=1}^{\infty} a_n \,.$$

(The absolute convergence of the rearranged series was proved in the previous theorem.)

Proof. Let $A := \sum\limits_{i=1}^{\infty} a_{p(i)}$. Furthermore, for any $n \in \mathbb{N}$ let

$$N = N(n) := \max\{p^{-1}(1),\, p^{-1}(2),\, \dots,\, p^{-1}(n)\}\,,$$

$$M = M(n) := \max\{p(1),\, p(2),\, \dots,\, p(N)\}\,,$$

$$R_n := \sum_{i=1}^{N} a_{p(i)} - \sum_{k=1}^{n} a_k \,.$$

Then $n \le N \le M$, and

$$\{1,\, 2,\, \dots,\, n\} \subseteq \{p(1),\, p(2),\, \dots,\, p(N)\} \subseteq \{1,\, 2,\, \dots,\, M\}\,,$$

therefore – using Theorem 2.11 – we have

$$|R_n| = \left| \sum_{i=1}^{N} a_{p(i)} - \sum_{k=1}^{n} a_k \right| \le \sum_{i=1}^{N} |a_{p(i)}| - \sum_{k=1}^{n} |a_k| \le$$
$$\le \sum_{k=1}^{M} |a_k| - \sum_{k=1}^{n} |a_k| = \sum_{k=n+1}^{M} |a_k| \,.$$

Since the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then – using Theorem 6.12 – we have

$$\lim_{n\to\infty} \sum_{k=n+1}^{M} |a_k| = \lim_{n\to\infty} \sum_{k=n+1}^{M(n)} |a_k| = 0\,.$$

Consequently $\lim\limits_{n\to\infty} R_n = 0$.

Therefore

$$\sum_{k=1}^{n} a_k = \sum_{i=1}^{N} a_{p(i)} - R_n \longrightarrow A - 0 = A \qquad (n \to \infty)\,.$$

$\square$

7.2. The Root Test and the Ratio Test

7.9. Theorem *[Root Test]*

Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N})$ *and suppose that the limit*

$$L := \lim_{n\to\infty} \sqrt[n]{|a_n|} \in [0\,,\,+\infty]$$

exists. Then

a) If $L < 1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is absolutely convergent*

b) If $L > 1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent. Moreover,* $\lim\limits_{n\to\infty} |a_n| = +\infty$.

Proof.

a) Suppose that $L < 1$. Let $q \in \mathbb{R}$, $0 \le L < q < 1$. Then $q - L > 0$, therefore

$$\exists N \in \mathbb{N}\ \forall n \ge N: \quad L - (q - L) < \sqrt[n]{|a_n|} < L + (q - L)\,.$$

The right side inequality implies that

$$0 \le \sqrt[n]{|a_n|} < q \qquad (n \ge N)\,.$$

Taking the n-th power we obtain

$$|a_n| < q^n \qquad (n \ge N)\,.$$

By $0 < q < 1$ the geometric series $\sum\limits_{n=N}^{\infty} q^n$ is convergent, then by the majorant criterion the series $\sum\limits_{n=1}^{\infty} |a_n|$ is also convergent.

b) Suppose that $L > 1$. Let $q \in \mathbb{R}$, $1 < q < L$. Then $L - q > 0$, therefore

$$\exists N \in \mathbb{N}\ \forall n \ge N: \quad L - (L - q) < \sqrt[n]{|a_n|} < L + (L - q)\,.$$

The left side inequality implies that

$$\sqrt[n]{|a_n|} > q \qquad (n \ge N)\,.$$

Taking the n-th power we obtain

$$0 \le |a_n| > q^n \qquad (n \ge N)\,.$$

This inequality shows us that (a_n) cannot be a zero sequence, consequently – by the zero sequence test – the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent. Moreover, $\lim\limits_{n\to\infty} |a_n| = +\infty$.

□

7.10. Remarks.

1. The theorem does not say anything about the case $L = 1$. This is the indeterminate case. In this case anything can happen. For example, at each of the following sequences $L = 1$, but

   a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent

   b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

   c) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the terms tend to 0.

   d) $\sum_{n=1}^{\infty} 1$ is divergent, the terms do not tend to 0, the terms form a bounded sequence

   e) $\sum_{n=1}^{\infty} n$ is divergent, the terms do not tend to 0, the terms form an unbounded sequence

2. The essentiality of the proof of the theorem was the application of the Direct Comparison Test with a geometric series. Thus it turns out from the proof of the root test that it works (i.e. $L \neq 1$) in those cases when the sequence of the absolute values of the terms $(|a_n|)$

   - either tends to 0 faster than a geometric sequence with some $0 \leq q < 1$,
   - or tends to ∞ faster than a geometric sequence with some $q > 1$.

   In all other cases the root test is inactive, that is it finishes in the indeterminate case $L = 1$.

3. If the statement in part b) is only the divergency of the series, then the proof is simpler if you start with $L - 1 > 0$ instead of $L - q > 0$.

We remark that the root test can be extended to the case when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ does not exist.

7.11. Theorem *[Ratio Test]*
Let $a_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N})$ *and suppose that the limit*

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0\,,\, +\infty]$$

exists. Then

a) If $L<1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is absolutely convergent*

b) If $L>1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent. Moreover,* $\lim\limits_{n\to\infty} |a_n| = +\infty$*.*

Proof.

a) Suppose that $L<1$. Let $q\in\mathbb{R}$, $0\le L<q<1$. Then $q-L>0$, therefore

$$\exists N\in\mathbb{N}\ \forall n\ge N: \quad L-(q-L)<\left|\frac{a_{n+1}}{a_n}\right|<L+(q-L)\,.$$

The right side inequality implies that

$$0\le\left|\frac{a_{n+1}}{a_n}\right|<q \qquad (n=N,\,N+1,\,N+2,\,\dots)\,.$$

Let us fix a natural number $k\ge N+1$, and write the above inequalities for $n=N\,,\ N+1\ \dots,\ k-1$. Then multiply these $k-N$ inequalities with each other. We obtain that

$$\left|\frac{a_{N+1}}{a_N}\right|\cdot\left|\frac{a_{N+2}}{a_{N+1}}\right|\cdot\left|\frac{a_{N+3}}{a_{N+2}}\right|\cdot\ldots\cdot\left|\frac{a_k}{a_{k-1}}\right|<\underbrace{q\cdot q\cdot\ \ldots\ \cdot q}_{k-N},$$

that is

$$\left|\frac{a_{N+1}\cdot a_{N+2}\cdot a_{N+3}\cdot\ldots\cdot a_k}{a_N\cdot a_{N+1}\cdot a_{N+2}\cdot\ldots\cdot a_{k-1}}\right|<q^{k-N}\,.$$

Hence – after simplifications – we have

$$\left|\frac{a_k}{a_N}\right|<\frac{q^k}{q^N}\,,$$

that is

$$|a_k|<\frac{|a_N|}{q^N}\cdot q^k \qquad (k\ge N+1)\,.$$

By $0<q<1$ the series

$$\sum_{k=1}^{\infty}\frac{|a_N|}{q^N}\cdot q^k=\frac{|a_N|}{q^N}\cdot\sum_{k=1}^{\infty}q^k$$

is convergent, then by the majorant criterion the series $\sum\limits_{n=1}^{\infty}|a_n|$ is also convergent.

b) Suppose that $L>1$. Let $q\in\mathbb{R}$, $1<q<L$. Then $L-q>0$, therefore

$$\exists N\in\mathbb{N}\ \forall n\ge N: \quad L-(L-q)<\left|\frac{a_{n+1}}{a_n}\right|<L+(L-q)\,.$$

The left side inequality implies that

$$\left|\frac{a_{n+1}}{a_n}\right| > q \qquad (n = N,\, N+1,\, N+2,\, \dots)\,.$$

As in part a), let us fix a natural number $k \geq N+1$, and write the above inequalities for $n = N\,,\ N+1\ \dots,\ k-1$. Then multiply these $k-N$ inequalities with each other. We obtain that

$$\left|\frac{a_{N+1}}{a_N}\right| \cdot \left|\frac{a_{N+2}}{a_{N+1}}\right| \cdot \left|\frac{a_{N+3}}{a_{N+2}}\right| \cdot \ldots \cdot \left|\frac{a_k}{a_{k-1}}\right| > \underbrace{q \cdot q \cdot \ \ldots\ \cdot q}_{k-N}\,,$$

that is

$$\left|\frac{a_{N+1} \cdot a_{N+2} \cdot a_{N+3} \cdot \ldots \cdot a_k}{a_N \cdot a_{N+1} \cdot a_{N+2} \cdot \ldots \cdot a_{k-1}}\right| > q^{k-N}\,.$$

Hence – after simplifications – we have

$$\left|\frac{a_k}{a_N}\right| > \frac{q^k}{q^N}\,,$$

that is

$$|a_k| > \frac{|a_N|}{q^N} \cdot q^k \qquad (k \geq N)\,.$$

This inequality shows us that (a_n) cannot be a zero sequence, consequently – by the zero sequence test – the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent. Moreover, $\lim\limits_{n\to\infty} |a_n| = +\infty$.

$\square$

7.12. Remark.

Writing „ratio test” instead of „root test”, then all the statements in Remarks 7.10 are true.

The ratio test can be extended to the case when $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

7.13. Remarks.

1. It was no coincidence that in the Root Test and in the Ratio Test we have used the same L. Later we will prove (see Corollary 8.12) that if both the limits

$$L_1 := \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{and} \quad L_2 := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exist, then $L_1 = L_2$.

2. We can give simple examples for that case when L_1 exists but L_2 does not exist. Let (a_n) be the following sequence:

$$a_{2n-1} := 1, \qquad a_{2n} := 2 \qquad (n \in \mathbb{N}) .$$

Then

$$\sqrt[2n-1]{|a_{2n-1}|} = \sqrt[2n-1]{1} = 1 \longrightarrow 1 \quad (n \to \infty) ,$$

$$\sqrt[2n]{|a_{2n}|} = \sqrt[2n]{2} = \sqrt[n]{\sqrt{2}} \longrightarrow 1 \quad (n \to \infty) .$$

Consequently $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$. The root test is inactive.

On the other hand,

$$\left|\frac{a_{2n}}{a_{2n-1}}\right| = \frac{2}{1} = 2 \longrightarrow 2 \quad (n \to \infty) ,$$

$$\left|\frac{a_{2n+1}}{a_{2n}}\right| = \frac{1}{2} = 1 \longrightarrow 1 \quad (n \to \infty) .$$

Consequently $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ does not exist.

Using the previous sequence (a_n) let

$$b_n := \frac{a_n}{2^n} \quad \text{and} \quad c_n := 2^n \cdot a_n \qquad (n \in \mathbb{N}) .$$

Then

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{2} = \frac{1}{2} < 1 ,$$

thus the root test is active, it shows that the series $\sum\limits_{n=1}^{\infty} b_n$ is convergent.

On the other hand

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} 2 \cdot \sqrt[n]{a_n} = 2 > 1 ,$$

thus the root test is active, it shows that the series $\sum\limits_{n=1}^{\infty} c_n$ is divergent.

However, the limits $\lim\limits_{n\to\infty} \left|\dfrac{b_{n+1}}{b_n}\right|$ and $\lim\limits_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right|$ do not exist.

3. It can be proved that if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists, then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ also exists.

7.3. Product of Series

If we want to multiply two finite sums $\sum\limits_{i=1}^{n} a_i$ and $\sum\limits_{j=1}^{m} b_j$, then – using the distributive law of numbers several times – we multiply every term by every term and then we add these partial products, whose number is mn. In formula:

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{j=1}^{m} b_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \,.$$

The result is independent of the order and grouping of the partial products at the addition on the right-hand side.

If we want to multiply two infinite sums $\sum\limits_{i=0}^{\infty} a_i$ and $\sum\limits_{j=0}^{\infty} b_j$, then the first step – in which we multiply every term by every term – results in the following „infinite by infinite" matrix

$$\begin{matrix} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix} \tag{7.1}$$

The ij-th entry of this matrix is equal to a_ib_j where $i, j \in \mathbb{N} \cup \{0\}$.

The problem is that the entries of this matrix form a double sequence (or: two-fold sequence), and we did not learn about the addition of the terms of such sequences. The result may depend on the order and grouping of the entries at the addition.

Unless investigating the theory of double sequences and series and their consequences for the product of series, we will discuss only a special but important way of addition of partial products in the above matrix.

7.14. Definition Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$. Then the series

$$\sum_{n=0}^{\infty} \sum_{\substack{i,j \in \mathbb{N}\cup\{0\} \\ i+j=n}} a_i b_j$$

is called the Cauchy product of the series $\sum\limits_{i=0}^{\infty} a_i$ and $\sum\limits_{j=0}^{\infty} b_j$.

The above formula often is written shortly as

$$\sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j \,.$$

7.15. Remarks.

1. The n-th term of the Cauchy product is

$$c_n := \sum_{i+j=n} a_i b_j = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 \,,$$

   which is the sum of the „n-th diagonal" of the infinite matrix (7.1).

   The sequence (c_n) is called the convolution of the sequences (a_n) and (b_n).

2. Another usual form of the n-th term of the Cauchy product is as follows:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \,.$$

3. The n-th partial sum S_n of the Cauchy product

$$S_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n} \sum_{i+j=k} a_i b_j$$

   can be rewritten in the following useful form:

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_i b_j \,.$$

   The equality of the two forms follows immediately from the equality of the summation index sets. Both the index sets sum up the elements of the left upper triangle with vertices $a_0 b_0$, $a_0 b_n$, $a_n b_0$ in the infinite matrix (7.1). The left-hand side index set sums by diagonals, the right-hand one sums by rows.

About the convergence of the Cauchy product we present the following two theorems. Their proofs use similar idea as Theorem 7.7 and Theorem 7.8.

7.16. Theorem *Let $a_n, b_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$. If the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are absolutely convergent, then their Cauchy product is absolutely convergent.*

Proof. By the absolutely convergence, the quantities

$$P := \sum_{n=0}^{\infty} |a_n| \qquad \text{and} \qquad Q := \sum_{n=0}^{\infty} |b_n|$$

are finite. If c_n denotes the n-th term of the Cauchy product, then for any $n \in \mathbb{N}$ holds

$$\sum_{k=0}^{n} |c_k| = \sum_{k=0}^{n} \left| \sum_{i+j=k} a_i b_j \right| \leq \sum_{k=0}^{n} \sum_{i+j=k} |a_i b_j| = \sum_{i=0}^{n} \sum_{j=0}^{n-i} |a_i b_j| \leq$$

$$\leq \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| = \left(\sum_{i=0}^{n} |a_i| \right) \cdot \left(\sum_{j=0}^{n} |b_j| \right) \leq P \cdot Q \,.$$

This result means that the partial sum sequence

$$\sum_{k=0}^{n} |c_k| \qquad (n \in \mathbb{N})$$

is bounded above, consequently $\sum\limits_{n=0}^{\infty} |c_n| < \infty$. □

7.17. Remark. It turns out from the proof, that the absolute convergence of the Cauchy product is true in the following stronger sense:

$$\sum_{n=0}^{\infty} \sum_{i+j=n} |a_i b_j| < \infty \,.$$

7.18. Theorem *Let $a_n,\, b_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$, and suppose that the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are absolutely convergent. Denote by A and by B their sum:*

$$A := \sum_{n=0}^{\infty} a_n \qquad \textit{and} \qquad B := \sum_{n=0}^{\infty} b_n \,.$$

Then the sum of their Cauchy product equals $A \cdot B$.

(The absolute convergence of the Cauchy product was proved in the previous theorem.)

Proof. As in the previous part, S_n denotes the n-th partial sum of the Cauchy product:

$$S_n = \sum_{i=0}^{n} \sum_{j=0}^{n-i} a_i b_j \qquad (n \in \mathbb{N} \cup \{0\}) \,.$$

We have to prove that $\lim\limits_{n\to\infty} S_n = AB$.

Let U_n and V_n be the following partial sums

$$U_n := \sum_{k=0}^{n} a_k \qquad \text{and} \qquad V_n := \sum_{k=0}^{n} b_k \,,$$

and let

$$R_n := U_n V_n - S_n \qquad (n \in \mathbb{N} \cup \{0\}) \,.$$

If we prove that $\lim\limits_{n\to\infty} R_n = 0$, then the proof is complete, because

$$S_n = U_n V_n - R_n \longrightarrow AB - 0 = AB \qquad (n \to \infty) \,.$$

To prove $\lim\limits_{n\to\infty} R_n = 0$ we will show that the complement subsequences (R_{2n}) and (R_{2n+1}) are zero sequences.

The proof of $\lim\limits_{n\to\infty} R_{2n} = 0$:

$$| R_{2n}| = | U_{2n}V_{2n} - S_{2n}| = \left| \left(\sum_{i=0}^{2n} a_i \right) \cdot \left(\sum_{j=0}^{2n} b_j \right) - \sum_{i=0}^{2n} \sum_{j=0}^{2n-i} a_i b_j \right| =$$

$$= \left| \sum_{i=0}^{2n} \sum_{j=0}^{2n} a_i b_j - \sum_{i=0}^{2n} \sum_{j=0}^{2n-i} a_i b_j \right| \leq \sum_{i=0}^{2n} \sum_{j=0}^{2n} |a_i b_j| - \sum_{i=0}^{2n} \sum_{j=0}^{2n-i} |a_i b_j|\,.$$

In the last step we applied Theorem 2.11.

Now we decrement the subtrahend in the following way:

$$\sum_{i=0}^{2n} \sum_{j=0}^{2n-i} |a_i b_j| \geq \sum_{i=0}^{n} \sum_{j=0}^{2n-i} |a_i b_j| \geq \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j|\,.$$

In the last step we used that $i \leq n$, consequently $2n - i \geq 2n - n = n$.

Consequently:

$$\begin{aligned}
| R_{2n}| &\leq \sum_{i=0}^{2n} \sum_{j=0}^{2n} |a_i b_j| - \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| = \\
&= \sum_{i=n+1}^{2n} \sum_{j=0}^{2n} |a_i b_j| + \sum_{i=0}^{n} \sum_{j=0}^{2n} |a_i b_j| - \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| = \\
&= \sum_{i=n+1}^{2n} \sum_{j=0}^{2n} |a_i b_j| + \sum_{i=0}^{n} \sum_{j=n+1}^{2n} |a_i b_j| = \\
&= \left(\sum_{i=n+1}^{2n} |a_i| \right) \cdot \left(\sum_{j=0}^{2n} |b_j| \right) + \left(\sum_{i=0}^{n} |a_i| \right) \cdot \left(\sum_{j=n+1}^{2n} |b_j| \right) \leq \\
&\leq Q \cdot \sum_{i=n+1}^{2n} |a_i| + P \cdot \sum_{j=n+1}^{2n} |b_j|\,.
\end{aligned}$$

The quantities P and Q are defined in the proof of the previous theorem.

Since the series $\sum\limits_{n=0}^{\infty} |a_n|$ and $\sum\limits_{n=0}^{\infty} |b_n|$ are convergent, then by Theorem 6.12

$$\lim_{n\to\infty} \sum_{i=n+1}^{2n} |a_i| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sum_{j=n+1}^{2n} |b_j| = 0\,,$$

therefore (R_{2n}) is really a zero sequence.

The proof of $\lim\limits_{n\to\infty} R_{2n+1} = 0$:

This proof is almost the same as in the previous case, only a few modification will be required.

$$
\begin{aligned}
|\,R_{2n+1}| = |\,U_{2n+1}V_{2n+1} - S_{2n+1}| &= \left| \left(\sum_{i=0}^{2n+1} a_i\right) \cdot \left(\sum_{j=0}^{2n+1} b_j\right) - \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1-i} a_i b_j \right| = \\
&= \left| \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1} a_i b_j - \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1-i} a_i b_j \right| \leq \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1} |a_i b_j| - \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1-i} |a_i b_j| \,.
\end{aligned}
$$

In the last step we applied Theorem 2.11.

Now we decrement the subtrahend in the following way:

$$
\sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1-i} |a_i b_j| \geq \sum_{i=0}^{n} \sum_{j=0}^{2n+1-i} |a_i b_j| \geq \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| \,.
$$

In the last step we used that $i \leq n$, consequently $2n+1-i \geq 2n+1-n = n+1 > n$.

Consequently:

$$
\begin{aligned}
|\,R_{2n+1}| &\leq \sum_{i=0}^{2n+1} \sum_{j=0}^{2n+1} |a_i b_j| - \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| = \\
&= \sum_{i=n+1}^{2n+1} \sum_{j=0}^{2n+1} |a_i b_j| + \sum_{i=0}^{n} \sum_{j=0}^{2n+1} |a_i b_j| - \sum_{i=0}^{n} \sum_{j=0}^{n} |a_i b_j| = \\
&= \sum_{i=n+1}^{2n+1} \sum_{j=0}^{2n+1} |a_i b_j| + \sum_{i=0}^{n} \sum_{j=n+1}^{2n+1} |a_i b_j| = \\
&= \left(\sum_{i=n+1}^{2n+1} |a_i|\right) \cdot \left(\sum_{j=0}^{2n+1} |b_j|\right) + \left(\sum_{i=0}^{n} |a_i|\right) \cdot \left(\sum_{j=n+1}^{2n+1} |b_j|\right) \leq \\
&\leq Q \cdot \sum_{i=n+1}^{2n+1} |a_i| + P \cdot \sum_{j=n+1}^{2n+1} |b_j| \,.
\end{aligned}
$$

From here follows – using Theorem 6.12 and similar argument as in the previous case – that (R_{2n+1}) is a zero sequence. $\square$

7.4. Homework

1. Determine whether the following series are convergent or not.

$$a)\ \sum_{n=1}^{\infty}\left(\frac{n+2}{2n}\right)^{n} \qquad b)\ \sum_{n=1}^{\infty}\frac{2^n\cdot n!}{n^n}$$

$$c)\ \sum_{n=1}^{\infty}\frac{n!}{7^n+45} \qquad d)\ \sum_{n=1}^{\infty}\frac{(\sqrt{2016})^n}{(2n+1)!}$$

2. Determine the Cauchy product of the series $\sum\limits_{n=0}^{\infty}\frac{1}{2^n}$ by itself. Using this result compute the sum of the series

$$\sum_{n=0}^{\infty}\frac{n}{2^n}\,.$$

8. Lesson 8

8.1. Function Series

8.1. Definition Let $D \neq \emptyset$ and $f_n : D \to \mathbb{K} \quad (n \in \mathbb{N})$ be a sequence of functions. The expression (designated sum)

$$f_1 + f_2 + f_3 + \cdots = \sum_{n=1}^{\infty} f_n$$

is called a function series. The functions f_n are the terms of the series. The function sequence

$$S_n := f_1 + f_2 + \ldots + f_n = \sum_{k=1}^{n} f_k \qquad (n \in \mathbb{N})$$

is called the partial sum sequence of the function series. S_n is the n-th partial sum.

8.2. Remarks.

1. If the common domain D is not given, then – by definition – D is the intersection of the domains of the functions f_n.

2. The addition of the functions in (S_n) is defined in the usual pointwise way:

$$S_n(x) := f_1(x) + \ldots + f_n(x) = \sum_{k=1}^{n} f_k(x) \qquad (x \in D;\ n \in \mathbb{N})\,.$$

3. The function series $\sum\limits_{n=1}^{\infty} f_n$ is often written using the symbol of its variable:

$$f_1(x) + f_2(x) + f_3(x) + \cdots = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in D)\,.$$

   In this sense we can speak about the function series $\sum\limits_{n=1}^{\infty} f_n(x)$ and say that x denotes its variable.

4. The starting index is not necessarily 1, it can be any integer. The starting index is frequently 0.

The convergency of a function series can be defined in a lot of senses. In our subject we will use what is called pointwise convergence.

8.3. Definition Let $\sum\limits_{n=1}^{\infty} f_n$ be a function series and $x \in D$. We say that this function series is convergent at the point x if the numerical series

$$\sum_{n=1}^{\infty} f_n(x)$$

is convergent. Otherwise we say that the function series is divergent at x.

8.4. Remark. Using the definition of convergence of a numerical series we obtain that the function series is convergent at x if and only if the numerical sequence $(S_n(x)\ n \in \mathbb{N})$ is convergent.

8.5. Definition Let $\sum\limits_{n=1}^{\infty} f_n$ be a function series. The set S of all $x \in D$ at which the function series is convergent is called its convergence set. In formula:

$$S := \left\{ x \in D \mid \sum_{n=1}^{\infty} f_n \quad \text{is convergent at } x \quad \right\} =$$
$$= \left\{ x \in D \mid \sum_{n=1}^{\infty} f_n(x) \quad \text{is convergent} \quad \right\} \subseteq D\,.$$

The function

$$f : S \to \mathbb{K}, \quad f(x) := \sum_{n=1}^{\infty} f_n(x) \qquad (x \in S)$$

is called the sum function (or simply the sum) of the function series.

If $\emptyset \neq T \subseteq S$, then we say that the function series is pointwise convergent on T. In this case the function $f_{|T}$ is called the sum function of the function series on T.

8.6. Example

Let

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_0(x) := 1, \quad f_n(x) := x^n \qquad (x \in \mathbb{R};\ n \in \mathbb{N})\,.$$

Then for any fixed $x \in \mathbb{R}$ the number sequence

$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} x^n$$

is a geometric series. Consequently, the convergence set of the function series $\sum\limits_{n=0}^{\infty} x^n$ is

$$S = \{x \in \mathbb{K} \mid |x| < 1\} = B(0,1)\,.$$

The sum function is

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad (x \in S = B(0,1))\,.$$

8.2. Power Series

8.7. Definition Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$ be a number sequence and let $x_0 \in \mathbb{K}$. The series

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n = a_0 + a_1 \cdot (x - x_0) + a_2 \cdot (x - x_0)^2 + \dots$$

is called a power series. The numbers a_n are the coefficients, the number x_0 is the centre of the power series. The symbol x is the variable of the power series.

8.8. Examples

1. $\sum\limits_{n=0}^{\infty} x^n$. Here $x_0 = 0$, $a_n = 1$.

2. $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$. Here $x_0 = 0$, $a_n = \dfrac{1}{n!}$.

3. $\sum\limits_{n=0}^{\infty} \dfrac{(x-5)^n}{n \cdot 2^n}$. Here $x_0 = 5$, $a_n = \dfrac{1}{n \cdot 2^n}$.

Now we will investigate the convergence set of a power series. Obviously, the power series is absolutely convergent at $x = x_0$ and its sum is equal to a_0, because in this case the sum contains only one term: a_0.

8.9. Theorem *Let* $\sum\limits_{n=0}^{\infty} a_n \cdot (x - x_0)^n$ *be a power series and denote by* S *its convergence set. Suppose that the following limit exists:*

$$L := \lim_{n \to \infty} \sqrt[n]{|a_n|} \in [0, +\infty] \, .$$

Then

a) If $L = 0$*, then the power series is absolutely convergent for any* $x \in \mathbb{K}$*. Thus* $S = \mathbb{K}$*.*

b) If $L = +\infty$*, then the power series is*

- *absolutely convergent at* $x = x_0$*,*
- *divergent at any* $x \neq x_0$*.*

Thus $S = \{x_0\}$*.*

c) If $0 < L < +\infty$*, then the power series is*

- *absolutely convergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| < \dfrac{1}{L}$*,*
- *divergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| > \dfrac{1}{L}$*.*

Thus $B(x_0, \frac{1}{L}) \subseteq S \subseteq \overline{B}(x_0, \frac{1}{L})$*.*

Proof. We have remarked that the power series is absolutely convergent at $x = x_0$, thus $x_0 \in S$. Suppose that $x \in \mathbb{K} \setminus \{x_0\}$ and apply the Root Test for the numerical series

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \,.$$

First we compute the limit of the n-th roots:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x-x_0)^n|} = \lim_{n\to\infty} \sqrt[n]{|a_n| \cdot |x-x_0|^n} = \lim_{n\to\infty} |x-x_0| \cdot \sqrt[n]{|a_n|} =$$
$$= |x-x_0| \cdot \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x-x_0| \cdot L \,.$$

Then we can discuss the cases of the theorem:

a) Suppose that $L = 0$. Then

$$|x-x_0| \cdot L = |x-x_0| \cdot 0 = 0 < 1 \,,$$

therefore by the Root Test the power series is absolutely convergent at x. This implies the statement of part a).

b) Suppose that $L = +\infty$. Then (here it is important that $x \neq x_0$)

$$|x-x_0| \cdot L = |x-x_0| \cdot (+\infty) = +\infty > 1 \,,$$

therefore by the Root Test the power series is divergent at x. This implies the statement of part b).

c) Suppose that $0 < L < +\infty$. Then

$$|x-x_0| \cdot L < 1 \quad \Longleftrightarrow \quad |x-x_0| < \frac{1}{L} \,.$$

This implies the statement of part c).

□

The convergence set of the power series can be investigated with the ratio test too. The following theorem can be proved – using the ratio test – as in the case of Theorem 8.9, therefore the proof will be omitted.

8.10. Theorem *Let* $\sum\limits_{n=0}^{\infty} a_n \cdot (x-x_0)^n$ *be a power series with coefficients* $a_n \neq 0$ $(n \in \mathbb{N} \cup \{0\})$, *and denote by* S *its convergence set. Suppose that the following limit exists:*

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \in [0, +\infty] \,.$$

Then

a) If $L = 0$, *then the power series is absolutely convergent for any* $x \in \mathbb{K}$. *Thus* $S = \mathbb{K}$.

b) If $L = +\infty$*, then the power series is*

- *absolutely convergent at* $x = x_0$*,*
- *divergent at any* $x \neq x_0$*.*

Thus $S = \{x_0\}$*.*

c) If $0 < L < +\infty$*, then the power series is*

- *absolutely convergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| < \dfrac{1}{L}$*,*
- *divergent at any* $x \in \mathbb{K}$ *for which holds* $|x - x_0| > \dfrac{1}{L}$*.*

Thus $B(x_0, \frac{1}{L}) \subseteq S \subseteq \overline{B}(x_0, \frac{1}{L})$*.*

8.11. Remark. It was no coincidence that in Theorem 8.9 and in Theorem 8.10 we have used the same L.

Suppose for a while that L_1 stands instead of L in Theorem 8.9 and that L_2 stands instead of L in Theorem 8.10. Suppose that $L_1 \neq L_2$, say $L_1 < L_2$, that is:

$$0 \leq L_1 < L_2 \leq +\infty\,.$$

Taking reciprocals we have:

$$0 \leq \frac{1}{L_2} < \frac{1}{L_1} \leq +\infty\,.$$

Here we agree that $\dfrac{1}{0} = +\infty$ and $\dfrac{1}{+\infty} = 0$.

Then taking an $x \in \mathbb{K}$ with $\dfrac{1}{L_2} < |x - x_0| < \dfrac{1}{L_1}$, we obtain that the power series is convergent and divergent at x at the same time. This is a contradiction. Therefore $L_1 = L_2$.

8.12. Corollary. If $a_n \in \mathbb{K} \setminus \{0\}$ $(n \in \mathbb{N} \cup \{0\})$ is a number sequence, and the limits

$$L_1 := \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{and} \quad L_2 := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exist, then $L_1 = L_2$.

8.13. Definition Using the foregoing notations, suppose that the limit L exists. Then the radius of convergence is defined as follows:

$$R := \begin{cases} +\infty & \text{if} \quad L = 0, \\[2ex] 0 & \text{if} \quad L = +\infty, \\[2ex] \dfrac{1}{L} & \text{if} \quad 0 < L < +\infty\,. \end{cases}$$

Shortly, $R = \dfrac{1}{L}$ if we agree that in this formula $\dfrac{1}{0} = +\infty$ and $\dfrac{1}{+\infty} = 0$.

8.14. Remarks.

1. Using the radius of convergence, we can shortly say that that the power series is absolutely convergent if $x \in B(x_0, R)$ and is divergent if $x \notin \overline{B}(x_0, R)$.

2. Theorem 8.9, Theorem 8.10 and the concept of radius of convergence can be generalized for the case when the limit L does not exist.

3. Suppose that for the radius of convergence R holds $0 < R < +\infty$.

   If $x \in \mathbb{K}$ and $|x - x_0| = R$, then nothing can be stated generally about the convergence at x (indeterminate case). For example, in the case $\mathbb{K} = \mathbb{R}$, at each of the following power series $R = 1$, but

   a) $\sum\limits_{n=1}^{\infty} x^n$ is divergent at $x = 1$ and at $x = -1$.

   b) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$ is divergent at $x = 1$ and is conditionally convergent at $x = -1$.

   c) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$ is absolutely convergent at $x = 1$ and at $x = -1$.

In the practice the most important power series are which have positive radius of convergence.

8.15. Definition Let $\sum\limits_{n=0}^{\infty} a_n \cdot (x - x_0)^n$ be a power series. Suppose that the radius of convergence is positive, that is $R > 0$. Then the function

$$f : B(x_0, R) \to \mathbb{K}, \qquad f(x) := \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n$$

is called the sum function of the power series.

8.16. Remark. The sum function of a power series is not exactly the same as the sum function of a function series, because the power series can converge at some points that are not in $B(x_0, R)$. For power series we will use the concept of sum function as written in the above definition.

8.3. Analytical Functions

8.17. Definition Suppose that the radius of convergence of a power series is positive. Then its sum function is called an analytical function.

8.18. Examples

1. The constant function $f(x) = c$ $(x \in \mathbb{K})$ (where $c \in \mathbb{K}$ is fixed) is analytical on $\mathbb{K}$, since

$$f(x) = c + 0x + 0x^2 + 0x^3 + \ldots \qquad (x \in \mathbb{K}), \qquad\qquad R = +\infty\,.$$

2. Every polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \qquad (x \in \mathbb{K})$$

is analytical on $\mathbb{K}$, since

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + 0x^{n+1} + 0x^{n+2} + \ldots \qquad (x \in \mathbb{K}), \qquad R = +\infty\,.$$

3. The function $f(x) := \dfrac{1}{1-x}$ $(x \in B(0,1))$ is analytical on $B(0,1)$, because it is the sum function of the power series $\sum\limits_{n=0}^{\infty} x^n$, and the radius of convergence of this power series is $R = 1 > 0$.

In what follows, we will prove an important inequality about the analytical functions. It will be basically important at the limits of analytical functions
(see: Theorem 13.3). We begin with an auxiliary theorem.

8.19. Theorem *[Auxiliary Theorem]*
Let $a_n \in \mathbb{K}$ $(n \in \mathbb{N} \cup \{0\})$ be a number sequence and let $x_0 \in \mathbb{K}$. Then the radii of convergence of the power series

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n \cdot a_n \cdot (x - x_0)^n$$

are equal.

Proof. For simplicity we will prove the theorem only in the case when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists or when $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists.

Denote by R the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n$$

and by R' the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n \cdot a_n \cdot (x - x_0)^n\,.$$

In the case when $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists, we have

$$R' = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{n \cdot |a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{n}} \cdot \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} \cdot R = R\,.$$

Similarly, in the case when $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ exists, we have

$$R' = \frac{1}{\lim\limits_{n\to\infty}\left|\frac{(n+1)a_{n+1}}{na_n}\right|} = \frac{1}{\lim\limits_{n\to\infty}\frac{n+1}{n}} \cdot \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{1}\cdot R = R\,.$$

□

After this preliminary we can state and prove the promised important inequality.

8.20. Theorem *Let $f \in \mathbb{K} \to \mathbb{K}$ be an analytical function defined by the power series*

$$f(x) := \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \qquad (x \in B(x_0, R))\,,$$

where the centre x_0 and the coefficients a_n lie in $\mathbb{K}$. Let $R > 0$ denote its radius of convergence. Then

$$\forall\, r \in \mathbb{R},\ 0 < r < R\ \exists\, M > 0\ \forall\, x, y \in B(x_0, r): \quad |\,f(x) - f(y)| \le M \cdot |\,x - y|\,.$$

Proof. For the sake of brevity let $h := x - x_0$ and $k := y - x_0$. Then $h - k = x - y$, and we have

$$\begin{aligned} f(x) - f(y) &= \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n - \sum_{n=0}^{\infty} a_n \cdot (y - x_0)^n = \\ &= \left(a_0 + \sum_{n=1}^{\infty} a_n \cdot h^n\right) - \left(a_0 + \sum_{n=1}^{\infty} a_n \cdot k^n\right) = \\ &= \sum_{n=1}^{\infty} a_n \cdot (h^n - k^n) = \sum_{n=1}^{\infty} a_n \cdot (h - k)(h^{n-1} + h^{n-2}k + \ldots, + k^{n-1}) = \\ &= (x - y) \cdot \sum_{n=1}^{\infty} a_n \cdot \sum_{i=0}^{n-1} h^{n-1-i} \cdot k^i\,. \end{aligned}$$

Since

$$|h| = |x - x_0| < r \quad \text{and} \quad |k| = |y - x_0| < r,$$

then

$$|h^{n-1-i}k^i| = |h|^{n-1-i}|k|^i < r^{n-1-i} \cdot r^i = r^{n-1} \qquad (i = 0, \ldots, n-1)\,.$$

Therefore

$$\begin{aligned} |\,f(x) - f(y)| = |x - y| \cdot \left|\sum_{n=1}^{\infty} a_n \cdot \sum_{i=0}^{n-1} h^{n-1-i}k^i\right| &\le |x - y| \cdot \sum_{n=1}^{\infty} |a_n| \cdot \sum_{i=0}^{n-1} |h^{n-1-i}k|^i < \\ < |x - y| \cdot \sum_{n=1}^{\infty} |a_n| \cdot \sum_{i=0}^{n-1} r^{n-1} &= |x - y| \cdot \sum_{n=1}^{\infty} |a_n| \cdot n \cdot r^{n-1} = \\ = |x - y| \cdot \frac{1}{r} \cdot \sum_{n=1}^{\infty} |a_n| \cdot n \cdot r^n &= |x - y| \cdot \frac{1}{r} \cdot \sum_{n=1}^{\infty} |na_n r^n|\,. \end{aligned}$$

If we prove that the number series

$$\sum_{n=1}^{\infty} n a_n r^n \tag{8.1}$$

is absolutely convergent, then we are ready, because the searched constant M in the statement will be:

$$M := \frac{1}{r} \cdot \sum_{n=1}^{\infty} |n a_n r^n| \,.$$

To prove that (8.1) is absolutely convergent let us take the number $x_0 + r \in \mathbb{K}$. Since

$$|(x_0 + r) - x_0| = |r| = r < R \,,$$

and using the Auxiliary Theorem, we obtain that the power series

$$\sum_{n=0}^{\infty} n \cdot a_n \cdot (x - x_0)^n$$

is absolutely convergent at $x = x_0 + r$. Thus the series

$$\sum_{n=0}^{\infty} n \cdot a_n \cdot ((x_0 + r) - x_0)^n = \sum_{n=0}^{\infty} n \cdot a_n \cdot r^n$$

is absolutely convergent. The proof is complete. $\square$

8.4. Homework

1. Determine the radius of convergence of the following power series. If it is positive, then give the domain of the corresponding analytical function.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2} \cdot x^n$ *b)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 3^n} \cdot (x-2)^n$

   c) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \cdot (x+3)^n$ *d)* $\displaystyle\sum_{n=0}^{\infty} \frac{3^n + (-2)^n}{n+1} \cdot (x+1)^n$

2. Expand the following functions into power series around the centre $x_0 = 0$, then around the centre $x_0 = 2$

   a) $\displaystyle f(x) = \frac{1}{1-x}$ *b)* $\displaystyle f(x) = \frac{3x}{1+x}$

   c) $\displaystyle f(x) = \frac{2x}{3-x}$ *d)* $\displaystyle f(x) = \frac{1}{3-2x}$

   e) $\displaystyle f(x) = \frac{x}{2+x}$

9. Lesson 9

9.1. Five Important Analytical Functions

In this section we will define five important analytical functions as the sum functions of everywhere absolutely convergent power series. First we give the definitions, after it we will state and partially prove the everywhere absolute convergence.

9.1. Definition Let us define the following functions.

a) The function $\exp : \mathbb{K} \to \mathbb{K}$,

$$\exp(x) := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{K})$$

is called the exponential function.

b) The function $\sin : \mathbb{K} \to \mathbb{K}$,

$$\sin(x) := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{K})$$

is called the sine (or: sinus) function.

c) The function $\cos : \mathbb{K} \to \mathbb{K}$,

$$\cos(x) := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \qquad (x \in \mathbb{K})$$

is called the cosine (or: cosinus) function.

d) The function $\sinh : \mathbb{K} \to \mathbb{K}$,

$$\sinh(x) := x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad (x \in \mathbb{K})$$

is called the hyperbolic sine (or: sinus hiperbolicus) function.

e) The function $\cosh : \mathbb{K} \to \mathbb{K}$,

$$\cosh(x) := 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad (x \in \mathbb{K})$$

is called the hyperbolic cosine (or: cosinus hiperbolicus) function.

The sin and the cos functions are called trigonometric, the sinh and the cosh functions are called hyperbolic functions respectively.

9.2. Theorem *All the five power series in the above definition are absolutely convergent at any $x \in \mathbb{K}$. Thus their radii of convergence are equal to $+\infty$.*

Proof.

a) We investigate the convergence of the power series in part a) immediately with the Ratio Test. If $x = 0$, then the series is absolutely convergent. If $x \in \mathbb{K} \setminus \{0\}$, then we write the quotient of the consecutive terms:

$$\left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \quad (n \to \infty)\,.$$

Since $0 < 1$, then the series is absolutely convergent.

b) In part b) also the Ratio Test will be applied. If $x = 0$, then the series is absolutely convergent. If $x \in \mathbb{K} \setminus \{0\}$, then the quotient of the consecutive terms is

$$\left| (-1)^{n+1} \cdot \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n \cdot x^{2n+1}} \right| = \frac{|x|^2}{(2n+2)(2n+3)} \to 0 \quad (n \to \infty)\,.$$

Since $0 < 1$, then the series is absolutely convergent.

The parts c), d), e) can be proved similarly.

□

9.3. Remarks.

1. It is obvious that if $x \in \mathbb{R}$, then $\exp(x) \in \mathbb{R}$, $\sin(x) \in \mathbb{R}$, $\cos(x) \in \mathbb{R}$, $\sinh(x) \in \mathbb{R}$, $\cosh(x) \in \mathbb{R}$. Thus the definitions of the five analytical functions is correct.

2. Using limits of functions it can be proved that the above defined real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is identical with the exponential function $x \mapsto e^x$ defined in secondary school.

3. Using integral calculus it can be proved that the above defined real trigonometric functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are identical with the trigonometric functions $x \mapsto \sin x$, $x \mapsto \cos x$ defined in secondary school.

9.4. Theorem *[The Simplest Properties of the Above defined Five Analytical Functions]*

a) $\exp 0 = 1,\ \sin 0 = 0,\ \cos 0 = 1,\ \sinh 0 = 0,\ \cosh 0 = 1$

b) For any $x \in \mathbb{K}$ hold:

$$\sin(-x) = -\sin(x) \qquad \cos(-x) = \cos(x)$$

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

Part b) means that sin *and* sinh *are odd functions,* cos *and* cosh *are even functions.*

Proof. a) In all the five cases we have computed the function values at the centre. In this case the power series is a one-term-sum, the single term is the first term of the series.

b)

$$\begin{aligned}\sin(-x) &= \sum_{n=0}^{\infty}(-1)^n \cdot \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{(-1)^{2n+1}x^{2n+1}}{(2n+1)!} = \\ &= \sum_{n=0}^{\infty}(-1)^n \cdot (-1) \cdot \frac{x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = -\sin(x)\,.\end{aligned}$$

The remainder equalities can be proved similarly. □

9.5. Theorem *Let i denote the imaginary unit in $\mathbb{C}$. Then for any $x \in \mathbb{K}$ holds*

a)

$$\exp(x) = \cosh(x) + \sinh(x) \tag{9.1}$$

b)

$$\exp(ix) = \cos(x) + i \cdot \sin(x) \qquad \textit{(Euler's identity)} \tag{9.2}$$

Proof.

a)

$$\exp x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \cosh(x) + \sinh(x).$$

b)

$$\begin{aligned}\exp(ix) &= \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = \sum_{n=0}^{\infty}\frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(ix)^{2n+1}}{(2n+1)!} = \\ &= \sum_{n=0}^{\infty}\frac{i^{2n}x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{i^{2n+1}x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(i^2)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{i(i^2)^n x^{2n+1}}{(2n+1)!} = \\ &= \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n}}{(2n)!} + i \cdot \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = \cos x + i \cdot \sin x \qquad (x \in \mathbb{K}).\end{aligned}$$

□

9.6. Corollary. a) Apply (9.1) for x:

$$\exp(x) = \cosh(x) + \sinh(x)\,,$$

and for $-x$:

$$\exp(-x) = \cosh(-x) + \sinh(-x) = \cosh(x) - \sinh(x)\,.$$

After addition and subtraction of these equalities we obtain:

$$\exp(x) + \exp(-x) = 2 \cdot \cosh(x) \quad \text{and} \quad \exp(x) - \exp(-x) = 2 \cdot \sinh(x)\,.$$

Thus we have proved

$$\cosh x = \frac{\exp x + \exp(-x)}{2} \quad \text{and} \quad \sinh x = \frac{\exp x - \exp(-x)}{2} \qquad (x \in \mathbb{K})\,.$$

b) Apply (9.2) for x:

$$\exp(ix) = \cos(x) + i \cdot \sin(x)\,,$$

and for $-x$:

$$\exp(-ix) = \cos(-x) + i \cdot \sin(-x) = \cos(x) - i \cdot \sin(x)\,.$$

After addition and subtraction of these equalities we obtain:

$$\exp(ix) + \exp(-ix) = 2 \cdot \cos(x) \quad \text{and} \quad \exp(ix) - \exp(-ix) = 2i \cdot \sin(x)\,.$$

Thus we have proved

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2} \quad \text{and} \quad \sin x = \frac{\exp(ix) - \exp(-ix)}{2i} \qquad (x \in \mathbb{K})\,.$$

9.7. Theorem *[Addition Formula of the Exponential Function]*
For any $x, y \in \mathbb{K}$ holds

$$\exp(x+y) = (\exp x) \cdot (\exp y)\,.$$

Proof. Apply the Cauchy product for the absolutely convergent power series of $\exp x$ and $\exp y$, then use the Binomial Theorem:

$$\begin{aligned}
\exp x \cdot \exp y &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} = \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k! \cdot (n-k)!} \cdot x^k \cdot y^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k} = \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (x+y)^n = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y)\,.
\end{aligned}$$

□

9.8. Corollary. Apply the Addition Formula for $y = -x$ where $x \in \mathbb{K}$. Then we obtain a formula for $\exp(-x)$:

$$1 = \exp 0 = \exp(x + (-x)) = (\exp x) \cdot (\exp(-x))\,,$$

whence

$$\exp(-x) = \frac{1}{\exp x} \qquad (x \in \mathbb{K})\,.$$

The above equations imply that $\forall\, x \in \mathbb{K}: \ \exp x \neq 0$, that is $0 \notin R_{\exp}$.

9.9. Theorem *[Addition Formulas of* cos sin cosh sinh*]*
For any $x, y \in \mathbb{K}$ holds

a) $\cos(x+y) = (\cos x)(\cos y) - (\sin x)(\sin y)$

b) $\sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y)$

c) $\cosh(x+y) = (\cosh x)(\cosh y) + (\sinh x)(\sinh y)$

d) $\sinh(x+y) = (\sinh x)(\cosh y) + (\cosh x)(\sinh y)$

Proof. We will prove part a). First we express cos with exp (Euler's Formula), then we apply the Addition Formula of exp. Finally, with the help of Euler's Formula we get back to sin and cos.

$$\cos(x+y) = \frac{\exp(i(x+y)) + \exp(-i(x+y))}{2} = \frac{\exp(ix+iy) + \exp((-ix)+(-iy))}{2} =$$
$$= \frac{(\exp(ix)) \cdot (\exp(iy)) + (\exp(-ix)) \cdot (\exp(-iy))}{2} =$$
$$= \frac{(\cos x + i\sin x)(\cos y + i\sin y) + (\cos x - i\sin x)(\cos y - i\sin y)}{2}.$$

Hence – completing the operations in the numerator – we obtain that the above fraction is equal to

$$\frac{2(\cos x)(\cos y) - 2(\sin x)(\sin y)}{2} = (\cos x)(\cos y) - (\sin x)(\sin y).$$

The proof of the other parts of the theorem is similar. □

9.10. Corollary.

1. If we apply the Addition Formulas in the previous theorem for $y = x$, then we obtain:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin(2x) = 2\sin x\cos x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\sinh(2x) = 2\sinh x\cosh x$$

2. If we apply the Addition Formulas of cos and cosh in the previous theorem for $y = -x$, then we obtain:

$$1 = \cos^2 x + \sin^2 x$$

$$1 = \cosh^2 x - \sinh^2 x$$

9.2. The Exponential Function and the Powers of e

We recall that the definition of e is

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

9.11. Theorem

$$\exp 1 = e .$$

Proof. By the definition of the exponential function

$$\exp 1 = \sum_{n=0}^{\infty} \frac{1}{n!} .$$

Denote by S_n the partial sums of this series:

$$S_n := \sum_{k=0}^{n} \frac{1}{k!} \qquad (n \in \mathbb{N}) .$$

Using the Binomial Theorem, let us see the following transformations for $n \in \mathbb{N},\ n \geq 2$:

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \cdot \left(\frac{1}{n}\right)^k \cdot 1^{n-k} = 1 + 1 + \sum_{k=2}^{n} \binom{n}{k} \cdot \frac{1}{n^k} = \\
&= 1 + 1 + \sum_{k=2}^{n} \frac{n \cdot (n-1) \cdot \ldots \cdot (n-(k-1))}{k!} \cdot \frac{1}{n^k} = \\
&= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \frac{n \cdot (n-1) \cdot \ldots \cdot (n-(k-1))}{n \cdot n \cdot \ldots \cdot n} = \\
&= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \underbrace{\left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right)}_{k-1 \text{ times}} .
\end{aligned} \tag{9.3}$$

Since the factors in the parentheses are less than 1, we obtain the following estimation:

$$\left(1 + \frac{1}{n}\right)^n < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{k-1 \text{ times}} = \sum_{k=0}^{n} \frac{1}{k!} = S_n \quad (n \geq 2)$$

Taking $n \to \infty$ we have:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \leq \lim_{n\to\infty} S_n = \exp 1 .$$

Thus we have proved that $e \le \exp 1$.

To prove the reverse inequality, let us fix a natural number $m \in \mathbb{N}$, $m \ge 2$ and write the identity (9.3) for $n > m$:

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) .$$

Then let us omit the terms with indices $k = m+1, \ldots, n$ from the right-hand sum. Thus the sum will be decreased.

$$\left(1 + \frac{1}{n}\right)^n > 1 + 1 + \sum_{k=2}^{m} \frac{1}{k!} \cdot \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) \qquad (n > m \ge 2) .$$

Since the number of terms in the sum is independent of n, then we can take the limit $n \to \infty$ by terms:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \ge$$

$$\ge 1 + 1 + \sum_{k=2}^{m} \frac{1}{k!} \cdot \left(\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)\right) \cdot \ldots \cdot \left(\lim_{n\to\infty} \left(1 - \frac{k-1}{n}\right)\right) \ge$$

$$\ge 1 + 1 + \sum_{k=2}^{m} \frac{1}{k!} \cdot \underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{k-1 \text{ times}} = \sum_{k=0}^{m} \frac{1}{k!} = S_m .$$

Thus we have for arbitrary $m \in \mathbb{N}$ $m \ge 2$ that

$$e \ge S_m .$$

Taking the limit $m \to \infty$ we obtain

$$e \ge \exp 1 .$$

□

9.12. Theorem *For any rational number $r \in \mathbb{Q}$ holds*

$$\exp r = e^r .$$

Proof. First let $p,\, q \in \mathbb{N}$ be positive integers. Applying two times the Addition Formula of exp we have

$$\left(\exp\left(\frac{p}{q}\right)\right)^q = \underbrace{\left(\exp\frac{p}{q}\right)\cdot\ \ldots\ \cdot\left(\exp\frac{p}{q}\right)}_{q \text{ times}} = \exp\left(\frac{p}{q}+\ \ldots\ +\frac{p}{q}\right) =$$

$$= \exp\left(q\cdot\frac{p}{q}\right) = \exp p = \exp(\underbrace{1+1+\ \ldots,\ +1}_{p \text{ times}}) =$$

$$= \underbrace{(\exp 1)\cdot\ \ldots\ \cdot(\exp 1)}_{p \text{ times}} = \underbrace{e\cdot e\cdot\ \ldots\ e}_{p \text{ times}} = e^p\,,$$

whence after extraction of a q-th root we obtain

$$\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}\,.$$

Thus the theorem is proved for $r \in \mathbb{Q},\ r > 0$. The case of negative rational numbers can be easily traced back to the proved case. Indeed, for any $r < 0$ holds:

$$\exp r = \exp(-(-r)) = \frac{1}{\exp(-r)} = \frac{1}{e^{-r}} = e^r\,.$$

We have used here that $-r > 0$ if $r < 0$.

Finally, the statement is trivial for $r = 0$, because

$$\exp 0 = 1 = e^0\,.$$

□

9.13. Remark. It can be shown – using the limits of functions – that the equality $\exp x = e^x$ is valid for any $x \in \mathbb{R}$ too.

9.3. The Irrational e

In the previous section (see Theorem 9.11) we have proved that

$$\sum_{n=0}^{\infty}\frac{1}{n!} = e\,.$$

First we prove an estimation for the speed of convergence of the above expansion.

9.14. Theorem *Denote by S_n the partial sums of the above series, that is*

$$S_n := \sum_{k=0}^{n}\frac{1}{k!} \qquad (n \in \mathbb{N})\,.$$

Then for any $n \in \mathbb{N}$ holds

$$0 < e - S_n < \frac{1}{n\cdot n!}\,.$$

Proof. S_n can be regarded as an infinite series for any fixed n :

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} a_k \ , \quad \text{where} \quad a_k = \begin{cases} \dfrac{1}{k!} & \text{if} \quad k \leq n \\ 0 & \text{if} \quad k > n \end{cases}$$

Hence we have

$$e - S_n = \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \left(\frac{1}{k!} - a_k \right) = \sum_{k=n+1}^{\infty} \frac{1}{k!} \, .$$

Thus $e - S_n > 0$. On the other hand, we can write that:

$$0 < e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots =$$

$$= \frac{1}{n!} \cdot \left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ \ldots \right) <$$

$$< \frac{1}{n!} \cdot \underbrace{\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ \ldots \right)}_{\text{geometric series}} =$$

$$= \frac{1}{n!} \cdot \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n!} \cdot \frac{1}{n+1-1} = \frac{1}{n \cdot n!} \, .$$

□

9.15. Theorem *Euler's number e is irrational.*

Proof. Suppose indirectly that e is rational. Then (using $e > 0$) there exist the positive integers p, $q \in \mathbb{N}$ such that $e = \dfrac{p}{q}$.

Apply the previous theorem for $n = q$:

$$0 < \frac{p}{q} - S_q < \frac{1}{q \cdot q!} \, .$$

Multiply these inequalities by $q\,!$:

$$0 < \frac{p}{q} \cdot q! - S_q \cdot q! < \frac{1}{q} < 1 \, . \tag{9.4}$$

Since

$$\frac{p}{q} \cdot q! = p \cdot (q-1)! \in \mathbb{Z} \, ,$$

and

$$S_q \cdot q! = q! + \frac{q!}{1!} + \frac{q!}{2!} + \frac{q!}{3!} + \ldots \frac{q!}{q!} \in \mathbb{Z}\,.$$

Thus $\frac{p}{q} \cdot q! - S_q \cdot q! \in \mathbb{Z}$. However, this is a contradiction with (9.4), because an integer never can lie in the open interval $(0,1)$. □

9.16. Remark. It can be proved that the number e is no root of any nonzero polynomial with rational coefficients, that is:

$$\forall\, f \in \mathbb{Q}[x] \setminus \{0\} : \quad f(e) \neq 0\,.$$

We say about this property of e that e is a transcendental number.

This fact has a very interesting linear algebraic consequence. We have seen in Linear Algebra that $\mathbb{R}$ is a vector space over the number field $\mathbb{Q}$. Let $n \in \mathbb{N}$ and let us see the vector system

$$1,\; e,\; e^2,\; \ldots\; e^n \in \mathbb{R}\,.$$

If we take a nontrivial linear combination with rational coefficients

$$\lambda_0 \cdot 1 + \lambda_1 \cdot e + \lambda_2 \cdot e^2 + \ldots \lambda_n \cdot e^n\,,$$

then it can be regarded $f(e)$ where f is the following polynomial in $\mathbb{Q}[x]$:

$$f(x) = \sum_{k=0}^{n} \lambda_k x^k\,.$$

Since e is transcendental, then $f(e) \neq 0$. This means that the vector system

$$1,\; e,\; e^2,\; \ldots\; e^n \in \mathbb{R}\,.$$

is linearly independent.

However, n is arbitrary, consequently $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$.

9.4. Homework

1. Prove the addition formulas for the functions sin, cosh, sinh:

   a) $\sin(x+y) = (\sin x)(\cos y) + (\cos x)(\sin y)$
   b) $\cosh(x+y) = (\cosh x)(\cosh y) + (\sinh x)(\sinh y)$
   c) $\sinh(x+y) = (\sinh x)(\cosh y) + (\cosh x)(\sinh y)$

   for any $x, y \in \mathbb{K}$.

2. Prove for any $x, y \in \mathbb{R}$ that

$$\exp(x + iy) = (\exp x) \cdot (\cos y + i \sin y)$$

where $i = \sqrt{-1}$. Using this result prove that

$$\forall\, z \in \mathbb{C} : \quad \exp z \neq 0\,.$$

3. Prove for any $x, y \in \mathbb{R}$ that

a) $\cos(x + iy) = (\cos x)(\cosh y) - i(\sin x)(\sinh y)$

b) $\sin(x + iy) = (\sin x)(\cosh y) + i(\cos x)(\sinh y)$

where $i = \sqrt{-1}$.

10. Lesson 10

10.1. Limits of Functions

The limit is the central concept of Mathematical Analysis. It expresses where the function value tends to if its variable tends to somewhere. In Analysis-1 we will investigate the limits of functions of type $\mathbb{R} \to \mathbb{R}$ (one variable functions).

Remember the different types of neighbourhoods:

a) The neighbourhoods of a finite number:

$$B(a,r) := \{x \in \mathbb{R} \mid |x-a| < r\} = (a-r,\, a+r) \subset \mathbb{R}\,.$$

b) The neighbourhoods of $+\infty$:

$$B(+\infty,r) := \{x \in \mathbb{R} \mid x > \frac{1}{r}\} = (\frac{1}{r},\, +\infty) \subset \mathbb{R}\,.$$

c) The neighbourhoods of $-\infty$:

$$B(-\infty,r) := \{x \in \mathbb{R} \mid x < -\frac{1}{r}\} = (-\infty,\, -\frac{1}{r}) \subset \mathbb{R}\,.$$

10.1. Definition (Accumulation Point and Isolated Point) Let $\emptyset \neq H \subseteq \mathbb{R}$ and $a \in \overline{\mathbb{R}}$. The point a is called an accumulation point of H if

$$\forall\, r > 0: \quad (B(a,r) \setminus \{a\}) \cap H \neq \emptyset\,.$$

The set of all accumulation points of H is denoted by H', that is

$$H' := \{a \in \overline{\mathbb{R}} \mid a \text{ is an accumulation point of } H\}\,.$$

The points of $H \setminus H'$ are called isolated points.

10.2. Remark. An accumulation point of H can be approximated from H with arbitrary accuracy, without using the accumulation point itself. The isolated point cannot be approximated in such way.

After these preliminaries it follows the definition of the limit:

10.3. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in D'_f$. We say that f has a limit at the point a if

$$\exists\, A \in \overline{\mathbb{R}}\ \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in (B(a,\delta) \setminus \{a\}) \cap D_f: \quad f(x) \in B(A,\varepsilon)\,.$$

As in the case of Theorem 3.11 it can be proved that A in this definition is unique. This unique A is called the limit of the function f at the point a. The following notations are used to express this fact:

$$A = \lim_a f, \quad A = \lim_{x \to a} f(x), \quad f(x) \to A\ (x \to a)\,.$$

10.4. Remarks.

1. Thus the fact $\lim\limits_a f = A$ can be expressed with neighbourhoods as
$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in (B(a,\delta) \setminus \{a\}) \cap D_f : \quad f(x) \in B(A,\varepsilon)\,.$$
2. We agree that „$\exists\ \lim\limits_a f = A$" contains the information that a is an accumulation point of D_f.

If we want to express $\lim\limits_a f = A$ with inequalities, we have to translate the above formula, using the different types of neighbourhoods. Since we have for a and for A 3-3 independent possibilities, then we have 9 possibilities for expressions of the limit with inequalities.

1. Limits at finite places
   (a) Finite limit at a finite place:
   $$\lim_{x\to a} f(x) = A \quad\Longleftrightarrow\quad \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ 0 < |x-a| < \delta : \quad |f(x)-A| < \varepsilon\,.$$
   (b) $+\infty$ limit at a finite place:
   $$\lim_{x\to a} f(x) = +\infty \quad\Longleftrightarrow\quad \forall\, P > 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ 0 < |x-a| < \delta : \quad f(x) > P\,.$$
   (c) $-\infty$ limit at a finite place:
   $$\lim_{x\to a} f(x) = -\infty \quad\Longleftrightarrow\quad \forall\, P < 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ 0 < |x-a| < \delta : \quad f(x) < P\,.$$
2. Limits at $+\infty$:
   (a) Finite limit at $+\infty$:
   $$\lim_{x\to +\infty} f(x) = A \quad\Longleftrightarrow\quad \forall\, \varepsilon > 0\ \exists\, R > 0\ \forall\, x \in D_f,\ x > R : \quad |f(x)-A| < \varepsilon\,.$$
   (b) $+\infty$ limit at $+\infty$:
   $$\lim_{x\to +\infty} f(x) = +\infty \quad\Longleftrightarrow\quad \forall\, P > 0\ \exists\, R > 0\ \forall\, x \in D_f,\ x > R : \quad f(x) > P\,.$$
   (c) $-\infty$ limit at $+\infty$:
   $$\lim_{x\to +\infty} f(x) = -\infty \quad\Longleftrightarrow\quad \forall\, P < 0\ \exists\, R > 0\ \forall\, x \in D_f,\ x > R : \quad f(x) < P\,.$$
3. Limits at $-\infty$:
   (a) Finite limit at $-\infty$:
   $$\lim_{x\to -\infty} f(x) = A \quad\Longleftrightarrow\quad \forall\, \varepsilon > 0\ \exists\, R < 0\ \forall\, x \in D_f,\ x < R : \quad |f(x)-A| < \varepsilon\,.$$

(b) $+\infty$ limit at $-\infty$:

$$\lim_{x\to-\infty} f(x) = +\infty \quad \Longleftrightarrow \quad \forall\, P>0\ \exists\, R<0\ \forall\, x\in D_f,\ x<R: \quad f(x)>P\,.$$

(c) $-\infty$ limit at $-\infty$:

$$\lim_{x\to-\infty} f(x) = -\infty \quad \Longleftrightarrow \quad \forall\, P<0\ \exists\, R<0\ \forall\, x\in D_f,\ x<R: \quad f(x)<P\,.$$

10.5. Examples

1. The constant function. Let $c\in\mathbb{R}$ be fixed and $f:\mathbb{R}\to\mathbb{R},\ f(x)=c$. Then for any $a\in\overline{\mathbb{R}}$ holds:
$$\lim_{x\to a} f(x) = \lim_{x\to a} c = c\,.$$
Indeed, let $\varepsilon>0$ be arbitrary. The any $\delta>0$ will be good, because
$$\forall\, x\in (B(a,\delta)\setminus\{a\})\cap D_f: \quad f(x)=c\in B(c,\varepsilon)\,.$$

2. The identity of $\mathbb{R}$. Let $f:\mathbb{R}\to\mathbb{R},\ f(x):=x$. Then for any $a\in\overline{\mathbb{R}}$ holds:
$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a\,.$$
Indeed, let $\varepsilon>0$ be arbitrary. The $\delta:=\varepsilon$ will be good, because
$$\forall\, x\in (B(a,\delta)\setminus\{a\})\cap D_f: \quad f(x)=x\in B(a,\delta)=B(a,\varepsilon)\,.$$

10.2. The Transference Principle

The Transference Principle makes a contact between the limit of functions and the limit of sequences.

10.6. Theorem *[Transference Principle]*
Let $f\in\mathbb{R}\to\mathbb{R}$, $a\in D_f'$ and $A\in\overline{\mathbb{R}}$. Then

$$\lim_{x\to a} f(x) = A \quad \Leftrightarrow \quad \forall\, x_n\in D_f\setminus\{a\}\quad (n\in\mathbb{N}),\ \lim x_n = a: \quad \lim f(x_n)=A\,.$$

A sequence (x_n) with the properties

$$x_n\in D_f\setminus\{a\}\quad (n\in\mathbb{N}), \qquad \lim_{n\to\infty} x_n = a$$

is called: allowed sequence (more precisely: allowed sequence of f with respect to a).

Proof. Assume first $\lim\limits_{x\to a} f(x)=A$.
Let (x_n) be an allowed sequence and let $\varepsilon>0$. Then by $\lim\limits_{x\to a} f(x)=A$ we have

$$\exists\,\delta>0\ \forall\, x\in (B(a,\delta)\setminus\{a\})\cap D_f: \quad f(x)\in B(A,\varepsilon)\,.$$

Since $\lim x_n = a$, then to $\delta > 0$

$$\exists\, N \in \mathbb{N}\ \forall\, n \geq N: \quad x_n \in (B(a,\delta) \setminus \{a\}) \cap D_f\,.$$

Combining the two observations we have

$$\exists\, N \in \mathbb{N}, \forall\, n \geq N: f(x_n) \in B(A,\varepsilon)\,.$$

Hence follows that $\lim\limits_{n\to\infty} f(x_n) = A$.

To prove the opposite direction suppose indirectly that $\lim\limits_{x\to a} f(x) = A$ is not true. Then

$$\exists\, \varepsilon > 0,\ \forall\, \delta > 0\ \exists\, x \in (B(a,\delta) \setminus \{a\}) \cap D_f: \quad f(x) \notin B(A,\varepsilon)\,. \tag{10.1}$$

Apply this statement with choosing $\delta := \dfrac{1}{n}$ $(n \in \mathbb{N})$. The x according to n will be denoted by x_n. Then (x_n) is an allowed sequence, because

$$x_n \in D_f \setminus \{a\}, \quad (n \in \mathbb{N}), \quad \text{and by} \quad |x_n - a| < \frac{1}{n} \to 0 \quad (n \to \infty)$$

holds $\lim\limits_{n\to\infty} x_n = a$.

Using the assumption of this direction of the theorem follows that $\lim\limits_{n\to\infty} f(x_n) = A$. This is a contradiction, because by (10.1):

$$f(x_n) \notin B(A,\varepsilon) \qquad (n \in \mathbb{N})\,.$$

□

10.3. Operations with Limits

Using the Transference Principle we can easily trace back the problem of operations with limits of functions into the problem of operations with limits of sequences.

Let us review the algebraic operations with real-valued functions.

10.7. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $g \in \mathbb{R} \to \mathbb{R}$, and suppose that $D_f \cap D_g \neq \emptyset$. Then

$f+g \in \mathbb{R} \to \mathbb{R}, \qquad D_{f+g} = D_f \cap D_g, \qquad (f+g)(x) = f(x)+g(x)$ the sum of f and g

$f-g \in \mathbb{R} \to \mathbb{R}, \qquad D_{f-g} = D_f \cap D_g, \qquad (f-g)(x) = f(x)-g(x)$ the difference of f and g

$fg \in \mathbb{R} \to \mathbb{R}, \qquad D_{fg} = D_f \cap D_g, \qquad (fg)(x) = f(x) \cdot g(x)$ the product of f and g

Suppose that $D := \{x \in D_f \cap D_g \mid g(x) \neq 0\} \neq \emptyset$. Then

$$\frac{f}{g} \in \mathbb{R} \to \mathbb{R}, \qquad D_{\frac{f}{g}} = D \qquad \frac{f}{g}(x) = \frac{f(x)}{g(x)} \qquad \text{the quotient of } f \text{ and } g$$

10.8. Theorem *Let us use the notations of the previous definition, and let $a \in \overline{\mathbb{R}}$. Then*

a) if $a \in D'_{f+g}$ then $\lim\limits_a (f+g) = \lim\limits_a f + \lim\limits_a g$;

b) if $a \in D'_{f-g}$ then $\lim\limits_a (f-g) = \lim\limits_a f - \lim\limits_a g$;

c) if $a \in D'_{fg}$ then $\lim\limits_a (fg) = (\lim\limits_a f) \cdot (\lim\limits_a g)$;

d) if $a \in D'_{\frac{f}{g}}$ then $\lim\limits_a \left(\dfrac{f}{g}\right) = \dfrac{\lim\limits_a f}{\lim\limits_a g}$;

in the following sense:

If the limits on the right-hand sides of the above equations exist, and the operations between them are defined, then the limits on the left-hand sides exist and they are equal to the expressions on the right-hand sides.

Proof. We will prove only the statement for the addition. The other rules can be proved similarly.

Let $A = \lim\limits_a f$ and $B = \lim\limits_a g$. We have to prove that $\lim\limits_a (f+g) = A + B$, provided $A + B$ is defined.

Let (x_n) be an allowed sequence of $f + g$ with respect to a. Then

$$x_n \in D_{f+g} \setminus \{a\} \quad (n \in \mathbb{N}), \qquad \lim_{n\to\infty} x_n = a\,.$$

By $D_{f+g} = D_f \cap D_g$ we deduce that (x_n) is an allowed sequence of f and of g respectively. Using the Transference Principle for f and for g we have:

$$\lim_{n\to\infty} f(x_n) = A \quad \text{and} \quad \lim_{n\to\infty} f(x_n) = B\,.$$

Using Theorem 5.23 about the limits of the sum of sequences we have:

$$\lim_{n\to\infty} (f+g)(x_n) = \lim_{n\to\infty} (f(x_n) + g(x_n)) = A + B\,.$$

Hence – using once more the Transference Theorem – follows that $\lim\limits_{x\to a} (f+g)(x) = A+B$. $\square$

10.9. Corollary. Applying the limit of the product in that case when one of the factors is constant, we obtain that

$$\lim_{x\to a} (c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)\,.$$

10.10. Examples

1. Using the limits of the identity function $x \to x$ and part c) of the previous theorem (limit of product) we have for any $n \in \mathbb{N}$:

$$\lim_{x \to +\infty} x^n = (\lim_{x \to +\infty} x)^n = (+\infty)^n = +\infty\,,$$

$$\lim_{x \to -\infty} x^n = (\lim_{x \to -\infty} x)^n = (-\infty)^n = (-1)^n \cdot (+\infty)\,,$$

$$\forall\, a \in \mathbb{R}: \quad \lim_{x \to a} x^n = (\lim_{x \to a} x)^n = a^n\,.$$

2. Using the previous result and applying part d) of the previous theorem (limit of quotient) we have

$$\lim_{x \to +\infty} \frac{1}{x^n} = \frac{1}{\lim\limits_{x \to +\infty} x^n} = \frac{1}{+\infty} = 0\,,$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = \frac{1}{\lim\limits_{x \to -\infty} x^n} = \frac{1}{(-1)^n \cdot (+\infty)} = 0\,.$$

3. Let $n \in \mathbb{N}$, a_0, a_1, $\ldots$, $a_n \in \mathbb{R}$, $a_n \neq 0$. Then the limits of the polynomial

$$P : \mathbb{R} \to \mathbb{R}, \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{k=0}^{\infty} a_k x^k$$

are as follows:

- If $a \in \mathbb{R}$, then

$$\lim_{x \to a} P(x) = \lim_{x \to a} (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0) =$$
$$= (a_n a^n + a_{n-1} a^{n-1} + \ldots + a_1 a + a_0) = P(a)\,.$$

- If $a = +\infty$ or $a = -\infty$, then let us see the following transformation:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = x^n \cdot \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)\,.$$

Hence

$$\lim_{x \to +\infty} P(x) = (\lim_{x \to +\infty} x^n) \cdot (a_n + 0 + \ldots + 0) = a_n \cdot (+\infty)\,,$$

and

$$\lim_{x \to -\infty} P(x) = (\lim_{x \to -\infty} x^n) \cdot (a_n + 0 + \ldots + 0) = a_n \cdot (-1)^n \cdot (+\infty)\,.$$

10.4. Homework

1. Prove by the definition of the limit that

$$a)\quad \lim_{x\to 1}\frac{x^2-9}{x^2-3x}=4 \qquad b)\quad \lim_{x\to 3}\frac{x^2-9}{x^2-3x}=2$$

$$c)\quad \lim_{x\to 2}\frac{x^2+4x-5}{x^3-1}=1 \qquad d)\quad \lim_{x\to 1}\frac{x^2+4x-5}{x^3-1}=2$$

$$e)\quad \lim_{x\to 2}\frac{x+1}{x^3-4x^2+4x}=+\infty \qquad f)\quad \lim_{x\to 1}\frac{2-3x}{x^3-2x^2+x}=-\infty$$

11. Lesson 11

11.1. One-sided Limits

In many cases the variable x approaches the number $a \in \mathbb{R}$ only from one direction, namely

- $x \to a$, but $x < a$: x approaches a from the left-hand side. This case can be extended for $a = +\infty$, but it does not give any newness with respect to the common limit, because $+\infty$ can be approached only from the left.
- $x \to a$, but $x > a$: x approaches a from the right-hand side. This case can be extended for $a = -\infty$, but it does not give any newness with respect to the common limit, because $-\infty$ can be approached only from the right.

In these cases we speak about left-hand limits and right-hand limits respectively. Their common names are: one-sided limits.

11.1. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in \overline{\mathbb{R}}$, $-\infty \le a < +\infty$. Suppose that a is an accumulation point of the set $(a, +\infty) \cap D_f$ (we say that a is a right-hand accumulation point of D_f). Then the right-hand limit of f at a is denoted by $\lim\limits_{a+} f$ and it is defined as follows:

$$\lim_{a+} f := \lim_{a} f_{|(a,+\infty)\cap D_f} .$$

11.2. Remarks.

1. In the case $a = -\infty$ the right-hand limit is equivalent to the common limit. Really, since
$$(-\infty, +\infty) \cap D_f = D_f ,$$
then the point $-\infty$ is or both right-hand accumulation point and accumulation point at the same time, or none of them. Furthermore, if it is, then:
$$\lim_{(-\infty)+} f = \lim_{-\infty} f .$$
2. Some other notations for the right-hand limit:
$$\lim_{x \to a+} f(x), \quad \lim_{\substack{x\to a\\ x>a}} f(x), \quad f(x) \to A \ (x \to a+), \quad f(x) \to A \ (x \to a,\ x > a) .$$
$$\lim_{a+0} f, \quad \lim_{x\to a+0} f(x), \quad f(x) \to A \ (x \to a+0), \quad f(a+0) .$$
3. We agree that „$\exists \lim\limits_{a+} f = A$" contains the information that a is a right-hand accumulation point of D_f.

4. Since the right-hand limit is a usual limit of the restricted function, then – with applicable modifications – the Transference Principle is valid for right-hand limits too.

5. Using the Transference Principle for right-hand limits, we can prove easily the connections between the algebraic operations and the right-hand limits.

At a finite place $a \in \mathbb{R}$ the right-hand limit can be expressed by inequalities as follows:

1. Finite right-hand limit:

$$\lim_{x\to a+} f(x) = A \quad \iff \quad \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ a < x < a+\delta : \quad |f(x)-A| < \varepsilon\,.$$

2. $+\infty$ right-hand limit:

$$\lim_{x\to a+} f(x) = +\infty \quad \iff \quad \forall\, P > 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ a < x < a+\delta : \quad f(x) > P\,.$$

3. $-\infty$ right-hand limit:

$$\lim_{x\to a+} f(x) = -\infty \quad \iff \quad \forall\, P < 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\ a < x < a+\delta : \quad f(x) < P\,.$$

11.3. Examples

1. Let $a \in \mathbb{R}$. Then

$$\lim_{x\to a+} \frac{1}{x-a} = +\infty\,.$$

Really, if $P > 0$, then $\delta := \dfrac{1}{P} > 0$ will be good in the definition, because

$$a < x < a+\delta \;\Rightarrow\; 0 < x-a < \delta = \frac{1}{P} \;\Rightarrow\; \frac{1}{x-a} > P\,.$$

2. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then – applying the previous result and the limit of product – we have

$$\lim_{x\to a+} \frac{1}{(x-a)^n} = \lim_{x\to a+} \left(\frac{1}{x-a}\right)^n = \left(\lim_{x\to a+} \frac{1}{x-a}\right)^n = (+\infty)^n = +\infty\,.$$

11.4. Definition Let $f \in \mathbb{R} \to \mathbb{R}$, Suppose that

Let $f \in \mathbb{R} \to \mathbb{R}$, $a \in \overline{\mathbb{R}}$, $-\infty < a \le +\infty$. Suppose that a is an accumulation point of the set $(-\infty,\, a) \cap D_f$ (we say that a is a left-hand accumulation point of D_f). Then the left-hand limit of f at a is denoted by $\lim\limits_{a-} f$ and it is defined as follows:

$$\lim_{a-} f := \lim_{a} f_{|(-\infty,\, a)\cap D_f}\,.$$

11.5. Remarks.

1. In the case $a = +\infty$ the left-hand limit is equivalent to the common limit. Really, since
$$(-\infty, +\infty) \cap D_f = D_f \,,$$
then the point $+\infty$ is or both left-hand accumulation point and accumulation point at the same time, or none of them. Furthermore, if it is, then:
$$\lim_{(+\infty)-} f = \lim_{+\infty} f \,.$$

2. Some other notations for the left-hand limit:
$$\lim_{x\to a-} f(x), \quad , \lim_{\substack{x\to a \\ x<a}} f(x) \quad f(x) \to A \ (x \to a-), \quad f(x) \to A \ (x \to a,\, x < a)\,.$$
$$\lim_{a-0} f, \quad \lim_{x\to a-0} f(x), \quad f(x) \to A \ (x \to a-0), \quad f(a-0)\,.$$

3. We agree that „$\exists \lim\limits_{a-} f = A$” contains the information that a is a left-hand accumulation point of D_f.

4. Since the left-hand limit is a usual limit of the restricted function, then – with applicable modifications – the Transference Principle is valid for left-hand limits too.

5. Using the Transference Principle for left-hand limits, we can prove easily the connections between the algebraic operations and the left-hand limits.

At a finite place $a \in \mathbb{R}$ the left-hand limit can be expressed by inequalities as follows:

1. Finite left-hand limit:
$$\lim_{x\to a-} f(x) = A \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 \ \exists\, \delta > 0 \ \forall\, x \in D_f,\ a-\delta < x < a : \quad |f(x)-A| < \varepsilon\,.$$

2. $+\infty$ left-hand limit:
$$\lim_{x\to a-} f(x) = +\infty \quad \Longleftrightarrow \quad \forall\, P > 0 \ \exists\, \delta > 0 \ \forall\, x \in D_f,\ a-\delta < x < a : \quad f(x) > P\,.$$

3. $-\infty$ left-hand limit:
$$\lim_{x\to a-} f(x) = -\infty \quad \Longleftrightarrow \quad \forall\, P < 0 \ \exists\, \delta > 0 \ \forall\, x \in D_f,\ a-\delta < x < a : \quad f(x) < P\,.$$

11.6. Examples

1. Let $a \in \mathbb{R}$. Then
$$\lim_{x\to a-} \frac{1}{x-a} = -\infty\,.$$
Really, if $P < 0$, then $\delta := -\dfrac{1}{P} > 0$ will be good in the definition, because
$$a-\delta < x < a \;\Rightarrow\; \frac{1}{P} = -\delta < x-a < 0 \;\Rightarrow\; \frac{1}{x-a} < P\,.$$

2. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then – applying the previous result and the limit of product – we have
$$\lim_{x\to a-} \frac{1}{(x-a)^n} = \lim_{x\to a-} \left(\frac{1}{x-a}\right)^n = \left(\lim_{x\to a-} \frac{1}{x-a}\right)^n = (-\infty)^n = (-1)^n \cdot (+\infty)\,.$$

3. Especially if n is an even number, then the right-hand and the left-hand limits of the previous function are equal (see the previous example and Examples 11.3). Thus we have:
$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty \qquad \text{if } n \text{ is even}\,.$$

Using the notations in the definitions of one-sided limits we can state the following theorems about the connection between the limits and the one-sided limits.

11.7. Theorem *If* $\lim\limits_{x\to a-} f(x) = \lim\limits_{x\to a+} f(x) = A$, *then* $\lim\limits_{x\to a} f(x) = A$.

Proof. Since $\exists \lim\limits_{x\to a-} f(x)$, then a is a left-hand accumulation point of D_f. This implies that a is an accumulation point of D_f.

Let $\varepsilon > 0$. Then by the definition of the left-hand limit we have:
$$\exists\, \delta_1 > 0\ \forall\, x \in B(a,\delta_1) \cap (-\infty,\, a) \cap D_f : \quad f(x) \in B(A,\varepsilon)\,.$$
Similarly, by the definition of the right-hand limit we have:
$$\exists\, \delta_2 > 0\ \forall\, x \in B(a,\delta_2) \cap (a,\, +\infty) \cap D_f : \quad f(x) \in B(A,\varepsilon)\,.$$
Let $\delta := \min\{\delta_1,\, \delta_2\} > 0$, and let us observe that
$$B(a,\delta) \cap (-\infty,\, a) \cap D_f \subseteq B(a,\delta_1) \cap (-\infty,\, a) \cap D_f$$
$$B(a,\delta) \cap (a,\, +\infty) \cap D_f \subseteq B(a,\delta_2) \cap (a,\, +\infty) \cap D_f$$
$$(B(a,\delta) \cap (-\infty,\, a) \cap D_f) \cup (B(a,\delta) \cap (a,\, +\infty) \cap D_f) = (B(a,\delta) \setminus \{a\}) \cap D_f\,.$$
This implies that:
$$\forall\, x \in B(a,\delta) \setminus \{a\}) \cap D_f : \quad f(x) \in B(A,\varepsilon)\,,$$
which means $\lim\limits_{x\to a} f(x) = A$. □

The reverse statement is stated in the following theorem, and it can be proved easily.

11.8. Theorem *Using the previous notations, suppose that* $\lim_{x\to a} f(x) = A$. *Then*

a) a is a left-hand accumulation point or a right-hand accumulation point of D_f*;*

b) If a is a left-hand accumulation point of D_f*, then* $\lim_{x\to a-} f(x) = A$*;*

c) If a is a right-hand accumulation point of D_f*, then* $\lim_{x\to a+} f(x) = A$.

11.2. Limits of Monotone Functions

11.9. Definition Let $f \in \mathbb{R} \to \mathbb{R}$. We say that f is

- monotonically increasing if $\forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) \le f(x_2)$
- strictly monotonically increasing if $\forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) < f(x_2)$
- monotonically decreasing if $\forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) \ge f(x_2)$
- strictly monotonically decreasing if $\forall\, x_1, x_2 \in D_f,\ x_1 < x_2 : \quad f(x_1) > f(x_2)$
- monotone if it is monotonically increasing or monotonically decreasing
- strictly monotone if it is strictly monotonically increasing or strictly monotonically decreasing

11.10. Remarks.

1. The strictly monotonically increasing functions are often called increasing functions or strictly increasing functions.
2. The strictly monotonically decreasing functions are often called decreasing functions or strictly decreasing functions.
3. The monotonically increasing functions are sometimes called nondecreasing functions.
4. The monotonically decreasing functions are sometimes called nonincreasing functions.

11.11. Theorem *Let* $f \in \mathbb{R} \to \mathbb{R}$ *a monotonically increasing function and let* $a \in \overline{\mathbb{R}}$.

a) If $a < +\infty$ *and* $a \in ((a, +\infty) \cap D_f)'$*, then*

$$\lim_{a+} f = \inf f[(a, +\infty) \cap D_f] = \inf\{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\}\,.$$

b) If $a > -\infty$ *and* $a \in ((-\infty, a) \cap D_f)'$*, then*

$$\lim_{a-} f = \sup f[(-\infty, a) \cap D_f] = \sup\{f(x) \in \mathbb{R} \mid x \in D_f,\ x < a\}\,.$$

Proof. We will prove only part a). The proof of part b) is similar.

Let

$$H := f[(a, +\infty) \cap D_f] = \{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\} \quad \text{and} \quad A := \inf H\,.$$

Obviously $-\infty \leq A < +\infty$. We have to prove that $\lim\limits_{a+} f = A$.

We distinguish four cases.

Case 1, $a = -\infty$ and $A = -\infty$:

Since $a = -\infty$, then $H = f[(-\infty, +\infty) \cap D_f] = f[D_f] = R_f$. We have to prove that $\lim\limits_{-\infty} f = -\infty$, that is

$$\forall\, P < 0\ \exists\, R < 0\ \forall\, x \in D_f,\, x < R: \quad f(x) < P\,. \tag{11.1}$$

To prove this, let $P < 0$. Since $A = -\infty$, then H is not bounded below. Thus

$$\exists\, x_0 \in D_f: \quad f(x_0) < P\,.$$

Let $R < \min\{0,\, x_0\}$. Then for any $x \in D_f$, $x < R$ holds $x < x_0$, consequently – by the monotonicity of f – we have

$$f(x) \leq f(x_0) < P\,.$$

Thus (11.1) is proved.

Case 2, $a = -\infty$ and $-\infty < A < +\infty$:

As in the previous case, $H = R_f$. We have to prove that $\lim\limits_{-\infty} f = A$, that is

$$\forall\, \varepsilon > 0\ \exists\, R < 0\ \forall\, x \in D_f,\, x < R: \quad |f(x) - A| < \varepsilon\,. \tag{11.2}$$

To prove this, let $\varepsilon > 0$. Since $A + \varepsilon$ is not a lower bound of H, then

$$\exists\, x_0 \in D_f: \quad f(x_0) < A + \varepsilon\,.$$

Let $R < \min\{0,\, x_0\} < 0$. Then for any $x \in D_f$, $x < R$ holds $x < x_0$, consequently – by the monotonicity of f – we have

$$f(x) \leq f(x_0) < A + \varepsilon\,.$$

However, A is a lower bound of H, therefore $f(x) \geq A$.

Summarizing these inequalities we have:

$$A - \varepsilon < A \leq f(x) \leq f(x_0) < A + \varepsilon\,,$$

which means $|f(x) - A| < \varepsilon$. Thus (11.2) is proved.

Case 3, $-\infty < a < +\infty$ and $A = -\infty$:

In this case $H = \{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\}$. We have to prove that $\lim\limits_{a+} f = -\infty$, that is

$$\forall\, P < 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\, a < x < a + \delta: \quad f(x) < P\,. \tag{11.3}$$

To prove this, let $P < 0$. Since $A = -\infty$, then H is not bounded below. Thus

$$\exists\, x_0 \in D_f,\, x_0 > a: \quad f(x_0) < P\,.$$

Let $\delta := x_0 - a > 0$. Then for any $x \in D_f$, $a < x < a + \delta = x_0$ holds $x < x_0$, consequently – by the monotonicity of f – we have

$$f(x) \leq f(x_0) < P\,.$$

Thus (11.3) is proved.

Case 4, $-\infty < a < +\infty$ and $-\infty < A < +\infty$:
In this case $H = \{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\}$. We have to prove that $\lim\limits_{a+} f = A$, that is

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \in D_f,\, a < x < a + \delta: \quad |f(x) - A| < \varepsilon\,. \tag{11.4}$$

To prove this, let $\varepsilon > 0$. Since $A + \varepsilon$ is not a lower bound of H, then

$$\exists\, x_0 \in D_f,\, x_0 > a: \quad f(x_0) < A + \varepsilon\,.$$

Let $\delta := x_0 - a > 0$. Then for any $x \in D_f$, $a < x < a + \delta = x_0$ holds $x < x_0$, consequently – by the monotonicity of f – we have

$$f(x) \leq f(x_0) < A + \varepsilon\,.$$

However, A is a lower bound of H, therefore $f(x) \geq A$.

Summarizing these inequalities we have:

$$A - \varepsilon < A \leq f(x) \leq f(x_0) < A + \varepsilon\,,$$

which means $|f(x) - A| < \varepsilon$. Thus (11.4) is proved. □

A similar theorem can be stated for monotonically decreasing functions. We tell it without proof.

11.12. Theorem *Let $f \in \mathbb{R} \to \mathbb{R}$ a monotonically decreasing function and let $a \in \overline{\mathbb{R}}$.*

a) If $a < +\infty$ and $a \in ((a, +\infty) \cap D_f)'$, then

$$\lim_{a+} f = \sup f[(a, +\infty) \cap D_f] = \sup\{f(x) \in \mathbb{R} \mid x \in D_f,\ x > a\}\,.$$

b) If $a > -\infty$ and $a \in ((-\infty, a) \cap D_f)'$, then

$$\lim_{a-} f = \inf f[(-\infty, a) \cap D_f] = \inf\{f(x) \in \mathbb{R} \mid x \in D_f,\ x < a\}\,.$$

11.13. Remark. Since the monotone sequences can be regarded as $f : \mathbb{N} \to \mathbb{R}$ monotone functions, and $\mathbb{N}' = \{+\infty\}$, then we obtain that the theorems about the limits of monotone sequences (see Theorem 5.4 and Theorem 5.19) are the special cases of the above theorems (Theorem 11.11 and Theorem 11.12).

11.3. Homework

1. Prove by the definitions of the one-sided limits that

$$a)\quad \lim_{x\to 2-} \frac{x^2-4}{|x-2|} = -4 \qquad b)\quad \lim_{x\to 2+} \frac{x^2-4}{|x-2|} = 4$$

$$c)\quad \lim_{x\to 3-} \frac{x-5}{x^2-2x-3} = +\infty \qquad d)\quad \lim_{x\to 3+} \frac{x-5}{x^2-2x-3} = -\infty$$

12. Lesson 12

12.1. Limits of Rational Functions at Infinity

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \qquad (x \in \mathbb{R}) \qquad \text{and}$$

$$Q(x) = b_m x^n + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0 \qquad (x \in \mathbb{R})$$

be two nonzero polynomials, where the coefficients are

$$a_0, a_1, \ldots, a_n \in \mathbb{R},\ a_n \neq 0 \quad \text{and} \quad b_0, b_1, \ldots, b_m \in \mathbb{R},\ b_m \neq 0\,.$$

In this section we investigate the limits of the rational function

$$\frac{P(x)}{Q(x)}$$

at $+\infty$ and at $-\infty$.

Let us see the following transformation:

$$\frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^n + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} = \frac{x^m \cdot \left(\dfrac{a_n}{x^{m-n}} + \ldots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}\right)}{x^m \cdot \left(b_m + \dfrac{b_{m-1}}{x} + \ldots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}\right)}\,.$$

Let us simplify by x^m and make up the limits. Then we have

$$\lim_{x\to+\infty} \frac{P(x)}{Q(x)} = \frac{\lim\limits_{x\to+\infty} \left(\dfrac{a_n}{x^{m-n}} + \ldots + \dfrac{a_0}{x^m}\right)}{b_m} = \begin{cases} \dfrac{a_n}{b_n} & \text{if} \quad m = n \\[2ex] \dfrac{a_n}{b_m} \cdot (+\infty) & \text{if} \quad m < n \\[2ex] 0 & \text{if} \quad m > n \end{cases}$$

and

$$\lim_{x\to-\infty} \frac{P(x)}{Q(x)} = \frac{\lim\limits_{x\to-\infty} \left(\dfrac{a_n}{x^{m-n}} + \ldots + \dfrac{a_0}{x^m}\right)}{b_m} = \begin{cases} \dfrac{a_n}{b_n} & \text{if} \quad m = n \\[2ex] \dfrac{a_n}{b_m} \cdot (-1)^{n-m} \cdot (+\infty) & \text{if} \quad m < n \\[2ex] 0 & \text{if} \quad m > n \end{cases}$$

12.2. Limits of Rational Functions at Finite Places

Let $P, Q \in \mathbb{R}[x] \setminus \{0\}$ and $a \in \mathbb{R}$. In this section we will investigate the limit

$$\lim_{x\to a} \frac{P(x)}{Q(x)}\,.$$

If $Q(a) \neq 0$, then the result is obvious:

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{\lim\limits_{x\to a} P(x)}{\lim\limits_{x\to a} Q(x)} = \frac{P(a)}{Q(a)}\,.$$

The interesting case is when a is a root of the denominator, that is $Q(a) = 0$. By Corollary 2.25 we can factor out from Q the root factor $x - a$ on the maximal degree. This means that we determine the number m and the polynomial Q_1 such that

$$Q(x) = (x-a)^m \cdot Q_1(x) \quad \text{where} \quad m \in \mathbb{N} \quad \text{and} \quad Q_1(a) \neq 0\,.$$

Then factor out from P the root factor $x - a$ on the maximal degree, but maximally m times. More precisely, we determine the number n and the polynomial P_1 such that

$$\text{or} \quad n = m \qquad \text{or} \quad (0 \leq n < m \quad \text{and} \quad P_1(a) \neq 0)\,.$$

12.1. Remarks.

1. If a is no root of P, then we are in the second case with $n = 0$, $P_1 = P$.
2. In the case $n = m$ it may be possible that the root factor can be factored out from P_1 several times, but this factoring out is unnecessary.

After these factorizations we will continue with two cases:

Case 1.: $n = m$.
In this case we have

$$\frac{P(x)}{Q(x)} = \frac{(x-a)^n P_1(x)}{(x-a)^n Q_1(x)} = \frac{P_1(x)}{Q_1(x)} \longrightarrow \frac{P_1(a)}{Q_1(a)} \qquad (x \to a)\,.$$

Case 2.: $0 \leq n < m$ and $P_1(a) \neq 0$.
In this case we have

$$\frac{P(x)}{Q(x)} = \frac{(x-a)^n P_1(x)}{(x-a)^m Q_1(x)} = \frac{1}{(x-a)^{m-n}} \cdot \frac{P_1(x)}{Q_1(x)} \longrightarrow \frac{P_1(a)}{Q_1(a)} \cdot \lim_{x\to a} \frac{1}{(x-a)^{m-n}} \qquad (x \to a)\,.$$

The limits of the function $x \to \dfrac{1}{(x-a)^{m-n}}$ can be determined using the Examples 11.3 and 11.6. Thus the result is

$$\lim_{x\to a+} \frac{P(x)}{Q(x)} = \frac{P_1(a)}{Q_1(a)} \cdot (+\infty)\,,$$

and

$$\lim_{x\to a-} \frac{P(x)}{Q(x)} = \frac{P_1(a)}{Q_1(a)} \cdot (-1)^{m-n} \cdot (+\infty)\,.$$

Especially if $m-n$ is even, then we have the common limit

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P_1(a)}{Q_1(a)} \cdot (+\infty)\,.$$

12.2. Remark. In the second case the product $\frac{P_1(a)}{Q_1(a)} \cdot (+\infty)$ is well-defined, because $P_1(a) \neq 0$.

12.3. Homework

1. Determine the following limits if they exist.

   a) $\displaystyle\lim_{x\to 2} \frac{x^2-5x+6}{x^2+5x-14}$ b) $\displaystyle\lim_{x\to 1} \frac{2x^3-3x^2+2x-1}{x^3+x^2-5x+3}$

   c) $\displaystyle\lim_{x\to 3} \frac{x^2+2x-15}{x^5-8x^4+16x^3+18x^2-81x+54}$

2. Determine the following limits if they exist.

   a) $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[5]{x}-1}$ b) $\displaystyle\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

   c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ d) $\displaystyle\lim_{x\to 0} \frac{5x}{\sqrt{1+x}-\sqrt{1-x}}$

13. Lesson 13

13.1. Limits of Analytical Functions at Finite Places

In Theorem 8.20 we have proved an important inequality for analytical functions. The essentiality of this statement was that for any analytical function

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \qquad (x \in B(x_0, R))$$

holds

$$|f(x) - f(y)| \leq M \cdot |x - y| \qquad (x, y \in B(x_0, r)) \,.$$

Here r is an arbitrary radius for which $0 < r < R$ holds, and the constant M may be depending at most on r. The statement is valid for real and for complex analytical functions too.

On the base of this inequality we can investigate the limits of analytical functions at finite places. Although we discuss the limits for functions of type $\mathbb{R} \to \mathbb{R}$, in the following theorem we will make an exception: we will state and prove the theorem for functions of type $\mathbb{K} \to \mathbb{K}$. For $\mathbb{K} = \mathbb{R}$ it will contain the $\mathbb{R} \to \mathbb{R}$ case.

Before the theorem the concept of finite limits at finite places will be extended for functions of type $\mathbb{K} \to \mathbb{K}$:

13.1. Definition Let $f \in \mathbb{K} \to \mathbb{K}$, $a \in \mathbb{K}$ and $A \in \mathbb{K}$. Suppose that a is an accumulation point of D_f, that is

$$\forall\, r > 0 : \quad (B(a, r) \setminus \{a\}) \cap D_f \neq \emptyset \,.$$

In this case we say that $\lim\limits_{x \to a} f(x) = A$ if

$$\forall\, \varepsilon > 0 \; \exists\, \delta > 0 \; \forall\, x \in (B(a, \delta) \setminus \{a\}) \cap D_f : \quad f(x) \in B(A, \varepsilon) \,,$$

or equivalently (using inequalities):

$$\forall\, \varepsilon > 0 \; \exists\, \delta > 0 \; \forall\, x \in D_f,\, 0 < |x - a| < \delta : \quad |f(x) - A| < \varepsilon \,.$$

13.2. Remark. In the case $\mathbb{K} = \mathbb{R}$ we have once more the definition of the finite limit at finite place for functions of type $\mathbb{R} \to \mathbb{R}$.

13.3. Theorem *Let $f \in \mathbb{K} \to \mathbb{K}$ be an analytical function:*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n \qquad (x \in B(x_0, R)) \,,$$

where R denotes the positive radius of convergence of its power series. Then for any $a \in B(x_0, R)$ holds

$$\lim_{x \to a} f(x) = f(a) \,.$$

Proof. Let us fix an arbitrary $a \in B(x_0, R)$. It is obvious that a is an accumulation point of $D_f = B(x_0, R)$. Furthermore

$$\exists\, r > 0 : \quad 0 \leq |a - x_0| < r < R \qquad \text{(for example} \quad r = \frac{R + |a - x_0|}{2}) \, .$$

Apply Theorem 8.20 with this r and with $y = a$. We obtain that

$$\exists\, M > 0 \;\forall\, x \in B(x_0, r) : \quad |f(x) - f(a)| \leq M \cdot |x - a| \, . \tag{13.1}$$

Let $\varepsilon > 0$. To this ε the following δ will be good:

$$\delta := \min \left\{ \frac{\varepsilon}{M}, \, r - |a - x_0| \right\} \, .$$

Really, if $0 < |x - a| < \delta$, then by

$$|x - x_0| = |x - a + a - x_0| \leq |x - a| + |a - x_0| < \delta + |a - x_0| \leq r - |a - x_0| + |a - x_0| = r$$

holds $x \in B(x_0, r)$. Consequently by (13.1) we have:

$$|f(x) - f(a)| \leq M \cdot |x - a| < M \cdot \delta \leq M \cdot \frac{\varepsilon}{M} = \varepsilon \, .$$

Thus by the definition $\lim\limits_{x \to a} f(x) = f(a)$. □

13.4. Examples

1. Using the above theorem it is obvious that for any $a \in \mathbb{K}$ hold

$$\lim_{x \to a} \exp x = \exp a, \qquad \lim_{x \to a} \sin x = \sin a, \qquad \lim_{x \to a} \cos x = \cos a \, ,$$

especially

$$\lim_{x \to 0} \exp x = 1, \qquad \lim_{x \to 0} \sin x = 0, \qquad \lim_{x \to 0} \cos x = 1 \, .$$

2. Let us discuss the basic $\dfrac{0}{0}$ type limit

$$\lim_{x \to 0} \frac{\sin x}{x} \, .$$

Applying the power series expansion of $\sin x$ we have for any $x \neq 0$:

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \, .$$

This power series is absolutely convergent for any $x \in \mathbb{K}$. Denote by $g : \mathbb{K} \to \mathbb{K}$ its sum function. The functions $x \mapsto \dfrac{\sin x}{x}$ and $x \mapsto g(x)$ differ only at 0, but it does not affect their limits at 0. Applying Theorem 13.3 for the analytical function g we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} g(x) = g(0) = 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \ldots = 1 \, .$$

Thus we have proved that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \, .$$

3. The following basic $\dfrac{0}{0}$ type limit will be
$$\lim_{x\to 0} \frac{1-\cos x}{x^2}\,.$$
Applying the power series expansion of $\cos x$ we have for any $x \neq 0$:
$$\begin{aligned}\frac{1-\cos x}{x^2} &= \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)}{x^2} = \\ &= \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots}{x^2} = \frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\ldots\,.\end{aligned}$$
This power series is absolutely convergent for any $x \in \mathbb{K}$. Denote by $g:\mathbb{K}\to\mathbb{K}$ its sum function. The functions $x \mapsto \dfrac{1-\cos x}{x^2}$ and $x \mapsto g(x)$ differ only at 0, but it does not affect their limits at 0. Applying Theorem 13.3 for the analytical function g we have
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} g(x) = g(0) = \frac{1}{2!}-\frac{0^2}{4!}+\frac{0^4}{6!}-\ldots = \frac{1}{2}\,.$$
Thus we have proved that
$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}\,.$$

4. Finally, let us determine the basic $\dfrac{0}{0}$ type limit
$$\lim_{x\to 0} \frac{\exp x - 1}{x}\,.$$
Applying the power series expansion of $\exp x$ we have for any $x \neq 0$:
$$\begin{aligned}\frac{\exp x-1}{x} &= \frac{1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots-1}{x} = \\ &= \frac{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots}{x} = \frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\end{aligned}$$
This power series is absolutely convergent for any $x \in \mathbb{K}$. Denote by $g:\mathbb{K}\to\mathbb{K}$ its sum function. The functions $x \mapsto \dfrac{\exp x-1}{x}$ and $x \mapsto g(x)$ differ only at 0, but it does not affect their limits at 0. Applying Theorem 13.3 for the analytical function g we have
$$\lim_{x\to 0} \frac{\exp x-1}{x} = \lim_{x\to 0} g(x) = g(0) = \frac{1}{1!}+\frac{0}{2!}+\frac{0^2}{3!}+\ldots = 1\,.$$
Thus we have proved that
$$\lim_{x\to 0} \frac{\exp x-1}{x} = 1\,.$$

13.2. Homework

1. Determine the following limits if they exist.

$$a)\quad \lim_{x\to 0} \frac{1-\cos 3x}{5x^2} \qquad\qquad b)\quad \lim_{x\to 0} \frac{\sin 6x - \sin 7x}{\sin 3x}$$

$$c)\quad \lim_{x\to 0} \frac{1+\sin x-\cos x}{1-\sin x-\cos x} \qquad\qquad d)\quad \lim_{x\to 0} \frac{1-\sqrt{\cos^3 x}}{1-\cos x}$$

$$e)\quad \lim_{x\to \frac{\pi}{6}} \frac{\sin\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-2\cos x} \qquad\qquad f)\quad \lim_{x\to \pi} \frac{\sin x}{\pi^2-x^2}$$