Discrete mathematics I

Complex numbers

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Extension of number sets

- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, \ldots\}$
  There is no natural number $x \in \mathbb{N}$ such that $x + 2 = 1$!
  On $\mathbb{N}$ subtraction is not defined for all numbers.
- **Integers:** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  In $\mathbb{Z}$ subtraction is always possible: $x = -1$.
  There is no integer $x \in \mathbb{Z}$ such that $x \cdot 2 = 1$!
  On $\mathbb{Z}$ division is not defined by all numbers.
- **Rational numbers:** $\mathbb{Q} = \left\{ \frac{p}{q} : \ p, q \in \mathbb{Z}, q \neq 0 \right\}$
  We can divide by any nonzero number in $\mathbb{Q}$: $x = \frac{1}{2}$.
  There is no rational number $x \in \mathbb{Q}$ such that $x^2 = 2$!
  Taking the square root of a rational number $\mathbb{Q}$ does not always produce a rational number, not even in the case of a nonnegative rational number.
- **Real numbers:** $\mathbb{R}$.
  We can take the square root of any nonnegative number in $\mathbb{R}$.
  There is no real number $x \in \mathbb{R}$ such that $x^2 = -1$!
  We cannot take the square root of negative numbers in $x \in \mathbb{R}$, since:
  $\forall x \in \mathbb{R} : x^2 \geq 0$.

Extension of number sets

Among complex numbers the equation $x^2 = -1$ can be solved!

Applications of complex numbers:

- solving equations;
- geometry;
- physics (fluid dynamics, quantum mechanics, relativity theory);
- computer graphics, quantum computers.

Introducing complex numbers

Definition (imaginary unit)

Let i be a solution to the equation $x^2 = -1$; i is called the imaginary unit.

We would like to extend the operations of addition and multiplication from the set of real numbers to a larger set containing i, while keeping the 'usual rules' of calculation and adding the rule: $i^2 = -1$. E.g.:

$$(1+i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$$

.

Definition of complex numbers (informal definition)

Definition (complex numbers)

The expressions of the form $a + bi$ where $a, b \in \mathbb{R}$, are called complex numbers with addition and multiplication defined as:

- addition: $(a + bi) + (c + di) = a + c + (b + d)i$.
- multiplication: $(a + bi)(c + di) = ac - bd + (ad + bc)i$.

The set of all complex numbers is denoted by $\mathbb{C}$. The form $a + bi$ where $a, b \in \mathbb{R}$ is called the algebraic form (or Cartesian or rectangular form) of a complex number.

Definition (real part and imaginary part of a complex number)

Let $z = a + bi$ $(a, b \in \mathbb{R})$ be a complex number. Then the real part of z is $Re(z) = a \in \mathbb{R}$ and the imaginary part of z is $Im(z) = b \in \mathbb{R}$.

- **Note:** $Im(z) \neq bi$
- The complex numbers of the form $a + 0 \cdot i$ are the real numbers. The complex numbers of the form $0 + bi$ are called pure imaginary numbers.
- Two complex numbers with algebraic forms $a + bi$ and $c + di$ are equal: $a + bi = c + di$, if and only if $a = c$ and $b = d$.

The definition of complex numbers (formal definition)

Definition (formal definition of complex numbers)

The set $\mathbb{C}$ of complex numbers is the set $\mathbb{R} \times \mathbb{R}$ together with the following operations:

- addition: $(a, b) + (c, d) = (a + c, d + b)$;
- multiplication: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

The two definitions of complex numbers are equivalent: $a + bi \leftrightarrow (a, b)$, e.g. $i \leftrightarrow (0, 1)$.
The format $a + bi$ is more convenient for manual calculations.
The format (a, b) is more convenient for use with computers.

Basic properties of addition and multiplication on $\mathbb{C}$

Based on the definitions it is easy to verify the following properties:

Proposition (Basic properties of operations on $\mathbb{C}$)

Properties of addition

1. *Associativity:* $\forall a, b, c \in \mathbb{C} : (a + b) + c = a + (b + c)$.
2. *Commutativity:* $\forall a, b \in \mathbb{C} : a + b = b + a$.
3. *Neutral element (zero element):* $\exists 0 \in \mathbb{C}$ (*zero element*) *such that*
$$\forall a \in \mathbb{C} : 0 + a = a + 0 = a.$$
4. *Additive inverse (opposite):* $\forall a \in \mathbb{C} : \exists -a \in \mathbb{C}$ (*opposite of a*) *such that*
$$a + (-a) = (-a) + a = 0.$$

Properties of multiplication

1. *Associativity:* $\forall a, b, c \in \mathbb{C} : (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. *Commutativity:* $\forall a, b \in \mathbb{C} : a \cdot b = b \cdot a$.
3. *Unit element:* $\exists 1 \in \mathbb{C}$ (*unit element*) *such that* $\forall a \in \mathbb{C} : 1 \cdot a = a \cdot 1 = a$.
4. *Multiplicative inverse (reciprocal):* $\forall a \in \mathbb{C} \setminus \{0\} : \exists a^{-1} = \frac{1}{a} \in \mathbb{C}$ (*reciprocal of a*) *such that* $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Distributivity: $\forall a, b, c \in \mathbb{C} : a(b + c) = ab + ac$ (*and* $(a + b)c = ac + bc$)

$(\mathbb{C},+,\cdot)$ is an algebraically closed field

Corollary:

- Because of the above properties, the algebraic structure $(\mathbb{C},+,\cdot)$ is a so called *field* (just like $(\mathbb{R},+,\cdot)$ and $(\mathbb{Q},+,\cdot)$).
- Informally we can say that we can calculate with complex numbers 'in the same way' as with real numbers (in sums and products we can 'move' the brackets; the order of the terms in a sum and of the factors in a product can be changed; brackets can be expanded by the distributive property etc.) with the additional rule: $i^2=-1$.

Fundamental Theorem of Algebra: It can also be shown – proof is not easy – that all polynomial equations of positive degree has solution in $\mathbb{C}$. Hence the field $(\mathbb{C},+,\cdot)$ is *algebraically closed*. ('No need to introduce further numbers!'):

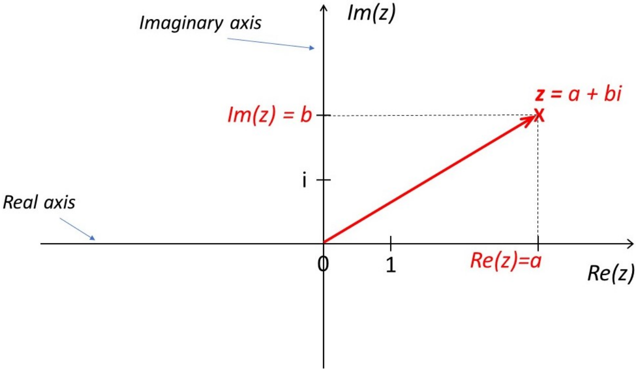
Theorem (Fundamental Theorem of Algebra; no proof required)

Let $n\in\mathbb{N}^+$. Then for every $a_0,\ldots,a_n\in\mathbb{C}$, $a_n\neq 0$, there exists $z\in\mathbb{C}$ such that $a_0+a_1z+a_2z^2+\ldots+a_nz^n=0$ (i.e. the polynomial $a_0+a_1x+a_2x^2+\ldots+a_nx^n$ has a root in $\mathbb{C}$.)

Representing complex numbers in the Complex plane (Gaussian plane, Argand diagram)

Complex numbers can be represented in the complex plane (Gaussian plane, Argand diagram):

- $z = a + bi \leftrightarrow (a, b)$
- bijecion (one-to-one correspondence) between $\mathbb{C}$ and the points (or vectors) of the plane.



Calculating with complex numbers: absolute value, conjugate

Definition (absolute value of a complex number)

The absolute value of a complex number z with algebraic form $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

In particular, if z is a real number, then $z = a$ and its absolute value is the 'usual' absolute value of a real number: $|z| = |a| = \sqrt{a^2}$.

Proposition (Hw)

For any complex number z:

1. $|z| \geq 0$,
2. $|z| = 0 \Leftrightarrow z = 0$.

Definition (conjugate of a complex number)

The conjugate of a complex number z with algebraic form $z = a + bi$ is $\overline{z} = a - bi$.

Calculating with complex numbers: opposite, subtraction

Definition (opposite of a complex number)

The opposite of a complex number z is the complex number denoted by $-z$ such that $z+(-z)=0$.

Proposition (Opposite of a complex number; proof is hw)

The opposite of a complex number z with algebraic form $z=a+bi$ is the complex number with algebraic form $-z=-a-bi$.

Definition (subtraction of complex numbers)

The difference of complex numbers z and w is defined as:

$$z-w=z+(-w)$$

Calculating with complex numbers: reciprocal, division

Definition (reciprocal of a nonzero complex number)

The reciprocal of a nonzero complex number z is the number $z^{-1} = \frac{1}{z}$ such that $z \cdot z^{-1} = 1$.

By the definition of multiplication it is easy to show that every nonzero complex number has a reciprocal.

Using the reciprocal, we can define division by nonzero complex numbers:

Definition (division by nonzero complex numbers)

The quotient of two complex numbers z and $w \neq 0$ is:

$$\frac{z}{w} = z \cdot \frac{1}{w}.$$

Calculating with complex numbers: reciprocal, division

What is $\frac{2+3i}{1+i}$ in algebraic form?

Idea: Similar to the rationalization of the denominator in fractions of real numbers:

$$\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1^2-\sqrt{2}^2} =$$

$$= \frac{1-\sqrt{2}}{1-2} = -1+\sqrt{2}$$

.

Multiply both the numerator and the denominator **by the conjugate of the denominator:**

$$\frac{2+3i}{1+i} = \frac{2+3i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{5+i}{1^2-i^2} = \frac{5+i}{1-(-1)} = \frac{5+i}{2} = \frac{5}{2} + \frac{1}{2}i$$

.

Why did this method work? When multiplying the denominator $1+i$ by its conjugate $1-i$, the result (the new denominator) is a real number.

Calculating with complex numbers: reciprocal, division

Lemma

For any complex number z we have $z \cdot \overline{z} = |z|^2$ (hence $z \cdot \overline{z}$ is a real number).

Proof

Let $z = a + bi$ be the algebraic form of z. Then
$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$.

Hence:

Proposition (Calculating the quotient in algebraic form)

Let $z, w \in \mathbb{C}$, $w \neq 0$. Then the quotient $\frac{z}{w}$ in algebraic form can be found as:

$$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}}$$

.

Proof

Let $z = a + bi$ and $w = c + di$ $(a, b, c, d \in \mathbb{R})$. Then
$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$.

Calculating with complex numbers

Theorem (Properties of conjugation and the absolute value of complex numbers; proof is hw.)

Let z and w be complex numbers. Then:

1. $\overline{\overline{z}} = z$;
2. $\overline{z+w} = \overline{z} + \overline{w}$;
3. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$;
4. $z + \overline{z} = 2Re(z)$;
5. $z - \overline{z} = 2Im(z) \cdot i$;
6. $z \cdot \overline{z} = |z|^2$;
7. *if* $z \neq 0$ *then* $z^{-1} = \frac{\overline{z}}{|z|^2}$;
8. $|0| = 0$ *and if* $z \neq 0$ *then* $|z| > 0$;
9. $|\overline{z}| = |z|$;
10. $|z \cdot w| = |z| \cdot |w|$;
11. $|z + w| \leq |z| + |w|$ *(triangle-inequality).*

Calculating with complex numbers

Theorem

$\vdots$

10. $|z \cdot w| = |z| \cdot |w|$;

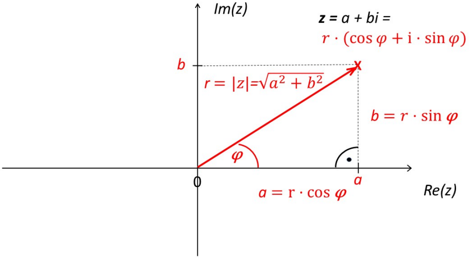
$\vdots$

Proof

$|z \cdot w|^2 = z \cdot w \cdot \overline{z \cdot w} = z \cdot w \cdot \overline{z} \cdot \overline{w} = z \cdot \overline{z} \cdot w \cdot \overline{w} = |z|^2 \cdot |w|^2 = (|z| \cdot |w|)^2.$

The polar form of complex numbers

Let $z = a + bi \in \mathbb{C}$ $(a, b \in \mathbb{R})$, $z \neq 0$.



- The length r of the vector (a, b) is: $r = \sqrt{a^2 + b^2} = |z|$.
- Denote by φ the angle from the positive real axis to the vector (a, b) (comment: this angle is not unique, because integer multiples of 2π can be added to it).

The coordinates a and b expressed in terms of r and φ ('polar coordinates'):

$$a = r \cdot \cos\varphi, \quad b = r \cdot \sin\varphi$$

The polar form of complex numbers

Definition (polar form)

The polar form of a nonzero complex number $z \in \mathbb{C}$ is:

$$z = r(\cos\varphi + i\sin\varphi)$$

where $r = |z|$.

Note:

- The polar form of zero is usually not used, because the angle could be any real number.
- The polar form is not unique (because the angle is not unique): $r(\cos\varphi + i\sin\varphi) = r(\cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi))$.

Definition (argument)

The argument of a nonzero $z \in \mathbb{C}$ is the angle $\varphi = arg(z) \in [0, 2\pi)$ such that $z = r(\cos\varphi + i\sin\varphi)$ where $r = |z|$.

Converting from algebraic form to polar form

Given the algebraic form $z = a + bi \neq 0$ we would like to determine the polar form of a nonzero complex number.

$$a + bi = r(\cos\varphi + i\sin\varphi)$$

Given a and b we are looking for $r = |z|$ and φ.

- Finding r: $r = |z| = \sqrt{a^2 + b^2}$.
- Finding φ: Since $a = r\cos\varphi$, hence

$$\varphi = \begin{cases} \arccos\frac{a}{r}, & \text{if } b \geq 0; \\ -\arccos\frac{a}{r}, & \text{if } b < 0. \end{cases}$$

De Moivre's formulas

Theorem (De Moivre's formulas)

Let $z, w \in \mathbb{C}$ be nonzero complex numbers: $z = |z|(\cos\varphi + i\sin\varphi)$, $w = |w|(\cos\psi + i\sin\psi)$, and let $n \in \mathbb{N}^+$. Then

1. $zw = |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi))$;
2. $\frac{z}{w} = \frac{|z|}{|w|} \cdot (\cos(\varphi-\psi) + i\sin(\varphi-\psi))$;
3. $z^n = |z|^n(\cos n\varphi + i\sin n\varphi)$.

The angles are added, subtracted, multiplied by n.

Geometric meaning

Multiplication by a nonzero complex number $z \in \mathbb{C}$ acts on the complex plane like an enlargement by a scale factor of $|z|$ together with a rotation by an angle of $arg(z)$ around the origin.

Proof

1.
$$\begin{aligned} zw &= |z|(\cos\varphi + i\sin\varphi)\cdot|w|(\cos\psi + i\sin\psi) = \\ &\quad \textit{By commutativity of multiplication:} \\ &= |z||w|(\cos\varphi + i\sin\varphi)\cdot(\cos\psi + i\sin\psi) = \\ &\quad \textit{By definition of multiplication:} \\ &= |z||w|(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\cos\varphi\sin\psi + \sin\varphi\cos\psi)) = \\ &\quad \textit{Hence by the trigonometric addition formulas:} \\ &= |z||w|(\cos(\varphi+\psi) + i\sin(\varphi+\psi)) \end{aligned}$$

Trigonometric addition formulas:
$\cos(\varphi+\psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi$
$\sin(\varphi+\psi) = \cos\varphi\sin\psi + \sin\varphi\cos\psi$

- The absolute value of the product: $|zw| = |z||w|$.
- The argument of the product:
  - if $0 \le arg(z) + arg(w) < 2\pi$ then $arg(zw) = arg(z) + arg(w)$;
  - if $2\pi \le arg(z) + arg(w) < 4\pi$ then $arg(zw) = arg(z) + arg(w) - 2\pi$.

  The functions sin, cos are periodic with a period 2π, for finding the argument of the product, we may need to reduce the sum of the arguments by 2π.

Roots of complex numbers

Definition (n^{th} roots of a complex number)

Let $n \in \mathbb{N}^+$ and $z \in \mathbb{C}$. The n^{th} roots of z are those complex numbers w for which $w^n = z$.

Theorem (Formula for the n^{th} roots of a complex number)

Let $z = |z|(\cos\varphi + i\sin\varphi)$, $n \in \mathbb{N}^+$. The n^{th} roots of z are:

$$w_k = \sqrt[n]{|z|}(\cos(\frac{\varphi}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\varphi}{n} + \frac{2k\pi}{n}))$$

$k = 0, 1, \ldots, n-1$.

The following fact will be used in the proof of the theorem:
Two complex numbers given in polar forms $z = |z|(\cos\varphi + i\sin\varphi)$ and $w = |w|(\cos\psi + i\sin\psi)$ are equal:

$$|z|(\cos\varphi + i\sin\varphi) = |w|(\cos\psi + i\sin\psi),$$

if and only if:

- $|z| = |w|$ and
- $\varphi = \psi + 2k\pi$ for some $k \in \mathbb{Z}$.

Roots of complex numbers

Theorem (Formula for the n^{th} roots of a complex number)

Let $z = |z|(\cos\varphi + i\sin\varphi)$, $n \in \mathbb{N}^+$. *The* n^{th} *roots of* z *are:*

$$w_k = \sqrt[n]{|z|}(\cos(\frac{\varphi}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\varphi}{n} + \frac{2k\pi}{n}))$$

$k = 0, 1, \ldots, n-1.$

Proof

By De Moivre's formula, for any complex number $w = |w|(\cos\psi + i\sin\psi)$ *we have* $w^n = |w|^n(\cos n\psi + i\sin n\psi)$.
Hence $w^n = z$ *is equivalent to* $|w|^n(\cos n\psi + i\sin n\psi) = |z|(\cos\varphi + i\sin\varphi)$, *which holds if and only if:*

- $|w|^n = |z| \Leftrightarrow |w| = \sqrt[n]{|z|}$ *and*
- $n\psi = \varphi + 2k\pi$ *for some* $k \in \mathbb{Z} \Leftrightarrow \psi = \frac{\varphi}{n} + \frac{2k\pi}{n}$ *for some* $k \in \mathbb{Z}$.

If $k \in \{0, 1, \ldots, n-1\}$, *then we obtain all distinct* n^{th} *roots.*

Example

Example

Find the 6^{th} roots (w) of $\frac{1-i}{\sqrt{3}+i}$.

$1-i=\sqrt{2}(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})=\sqrt{2}(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4})$

$\sqrt{3}+i=2(\frac{\sqrt{3}}{2}+i\frac{1}{2})=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

Since $\frac{7\pi}{4}-\frac{\pi}{6}=\frac{19\pi}{12}$, hence: $\frac{1-i}{\sqrt{3}+i}=\frac{1}{\sqrt{2}}(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12})$.

So the 6^{th} roots are:

$w_k=\frac{1}{\sqrt[12]{2}}(\cos\frac{19\pi+24k\pi}{72}+i\sin\frac{19\pi+24k\pi}{72})\ :\ k=0,1,\ldots,5$

Complex roots of unity

Definition (n^{th} roots of unit)

For any $n \in \mathbb{N}^+$ the n^{th} roots of 1 are called the n^{th} roots of unity. (i.e. the complex numbers ε satisfying $\varepsilon^n = 1$.)

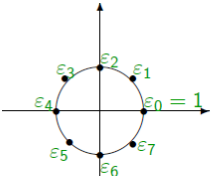
Using the formula of the n^{th} roots of a complex number we obtain the following:

Theorem (The polar form of the n^{th} roots of unity)

For any $n \in \mathbb{N}^+$ the n^{th} roots of unity are:

$$\varepsilon_k = \varepsilon_k^{(n)} = (\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}) \ : \ k = 0, 1, \ldots, n-1.$$

The 8^{th} roots of unity:



Roots of complex numbers

Theorem (Expressing all n^{th} roots of a complex number using one n^{th} root and the n^{th} roots of unity)

Let $z \in \mathbb{C}$ be a nonzero complex number, $n \in \mathbb{N}^+$ and $w \in \mathbb{C}$ be such that $w^n = z$. Then the n^{th} roots of z can be expressed in the following form:

$$w_k = w\varepsilon_k^{(n)} \text{ where } k = 0, 1, \ldots, n-1.$$

Proof

All numbers of the form $w\varepsilon_k$ are n^{th} roots of z:
$(w\varepsilon_k)^n = w^n\varepsilon_k^n = z \cdot 1 = z$. *These are n distinct values, hence we have obtained all n^{th} roots of z.*

Order

Definition (order of a complex number)

The order of a complex number $z \neq 0$, denoted by $o(n)$, is the smallest $n \in \mathbb{N}^+$ such that $z^n = 1$, if such an $n \in \mathbb{N}^+$ exists, otherwise it is defined as ∞.

- $1, 1, 1, \ldots \Rightarrow o(1) = 1$
- $-1, 1, -1, 1, \ldots \Rightarrow o(-1) = 2$
- $i, -1, -i, 1, i, -1, \ldots \Rightarrow o(i) = 4$
- $\frac{1+i}{\sqrt{2}}, i, \frac{-1+i}{\sqrt{2}}, -1, \frac{-1-i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}}, 1, \frac{1+i}{\sqrt{2}}, i, \ldots \Rightarrow o(\frac{1+i}{\sqrt{2}}) = 8$

Example

- The order of 1 is 1;
- The order of -1 is 2: $-1, 1, \ldots$;
- The order of i is 4: $i, -1, -i, 1, \ldots$;
- The order of 2 is ∞ : $2, 4, 8, 16, \ldots$.

Order

Theorem (The properties of the order of complex numbers)

Let $z \neq 0$ be a complex number. Then:

1. *If $o(z) = \infty$ then the powers of z to any two distinct positive integer exponents are always distinct.*
2. *If $o(z)$ is finite, then the sequence of powers of z to positive integer exponents is periodic with a period $o(z)$, which means that for any $k, l \in \mathbb{N}^+$ we have $z^k = z^l \Leftrightarrow o(z) | k - l$. In particular $z^k = 1 \Leftrightarrow o(z) | k$.*

The proof of the above theorem is easy, but not required for the exam.

Primitive n^{th} roots of unity

The order of an n^{th} root of unity is not necessarily equal to n:

4^{th} roots of unity: $1, i, -1, -i$.

- the order of 1 is 1;
- the order of -1 is 2;
- the order of i is 4.

Definition (primitive n^{th} roots of unity)

If the order of an n^{th} root of unity is equal to n, then we call it a primitive n^{th} root of unity.

Two corollaries of the Theorem about the Properties of the order:

Corollary

- *If ε is a primitive n^{th} root of unity, then the list $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{n-1}$ is a list of all n^{th} roots of unity.*
- *A primitive n^{th} root of unity is a k^{th} root of unity if and only if $n|k$.*

Polar forms of the primitive n^{th} roots of unity

Example

- Primitive 1. root of unity: 1;
- Primitive 2. roots of unity: -1;
- Primitive 3. roots of unity: $\frac{-1 \pm i\sqrt{3}}{2}$;
- Primitive 4. roots of unity: $\pm i$;
- Primitive 5. roots of unity: $\dots$ (HW)
- Primitive 6. roots of unity: $\frac{1 \pm i\sqrt{3}}{2}$.

Proposition (Polar forms of the primitive n^{th} roots of unity; no proof required)

An n^{th} root of unity $\cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$ is a primitive n^{th} root of unity if and only if $\gcd(n,k) = 1$.