Discrete mathematics I

Combinatorics

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Combinatorics

The goal of combinatorics:

- To organize of elements of finite sets;
- To enumerate the different possible arrangements.

Examples:

- Among any eight people, there are always two who were born on the same day of the week.
- At least how many people do we need to have in a group in order to be certain that two of them have their birthdays on the same day of the year?
- At least how many people do we need to have in a group in order to be certain that two of them were be born in the same month?
- What is the number of all the possible car registration plates /telephone numbers / IP addresses?
- At least how many tickets do we have to complete in order to certainly win the jackpot in the lottery? (In the lottery five numbers from among the numbers $1-90$ are drawn randomly. You can bet by crossing out five from among the numbers $1-90$ on a lottery ticket. You win the jackpot if you crossed exactly those numbers which are drawn later.)

Addition principle (Rule of sum)

Addition principle

Given two finite, disjoint sets:

$$A = \{a_1, a_2, \dots, a_n\}, \quad B = \{b_1, b_2, \dots, b_m\}$$

in how many ways can we choose one element from A or B?

The possible choices: $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m$.

The number of possible choices: $n + m$.

Example

In a pastry shop, they have 3 kinds of sweet pastries (jam roll, cheese cake, coconut cube) and 2 kinds of savory pastries (scones, pretzels). In how many different ways can we choose one sweet or one savory pastry?
Solution: $3 + 2 = 5$.

Multiplication principle (Rule of product)

Multiplication principle

Given two finite, disjoint sets:

$$A = \{a_1, a_2, \ldots, a_n\}, \qquad B = \{b_1, b_2, \ldots, b_m\}$$

in how many ways can we choose one element from A and B?

$$\begin{array}{c|cccc} & b_1 & b_2 & \ldots & b_m \\ \hline a_1 & (a_1, b_1) & (a_1, b_2) & \ldots & (a_1, b_m) \\ a_2 & (a_2, b_1) & (a_2, b_2) & \ldots & (a_2, b_m) \\ \vdots & & & \ddots & \\ a_n & (a_n, b_1) & (a_n, b_2) & \ldots & (a_n, b_m) \end{array}$$

The number of possible choices: $n \cdot m$.

Example

In a pastry shop, they have 3 kinds of sweet pastries and 2 kinds of savory pastries. In how many different ways can we choose one sweet and one savory pastry? Solution: $3 \cdot 2 = 6$.

Summary: six basic arrangement types

Permutations without repetition: $n!$, a sequence of n distinct elements, containing each element exactly once (order matters, each element occurs exactly once).

Permutations with repetition: $\dfrac{(k_1 + k_2 + \ldots + k_m)!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$, a sequence of length $n = k_1 + k_2 + \ldots + k_m$, containing the element of kind i exactly k_i times ($1 \leq i \leq m$) (order matters, an element can occur more than once: the i^{th} kind of element occurs exactly k_i times).

Variations without repetition: $n!/(n-k)!$, a sequence of length k chosen from n different elements, containing each element at most once (order matters, an element can occur at most once).

Variations with repetition: n^k, a sequence of length k chosen from n different elements (order matters, an element can occur more than once).

Combinations without repetition: $\binom{n}{k}$, k-element subset of an n-element set (order does not matter, an element can occur at most once).

Combinations with repetition: $\binom{n+k-1}{k}$, choosing k times from among n elements disregarding the order in which the elements were chosen and allowing for choosing an element more than once (order does not matter, an element can occur more than once).

Permutations without repetition

Definition (permutation (without repetition))

A permutation (without repetition) of a finite set A is a sequence containing each element of A exactly once. (In other words, a possible ordering of the elements of A.)

Equivalent definition: A permutation of a set A is a bijection $A \to A$.

Definition (n factorial)

Let $n \in \mathbb{N}$. Then n factorial $n!$ is defined as
$n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$ if $n > 0$ and as $0! = 1$ if $n = 0$.

Theorem (Number of permutations)

Let $n \in \mathbb{N}$. The number of permutations of an n-element set is: $P_n = n!$.

Proof

The first element of the sequence can be chosen in n different ways. After this, the second element of the sequence can be chosen in $n-1$ different ways, ... Hence the number of permutations is $n(n-1) \cdot \ldots \cdot 2 \cdot 1$. □

Permutations without repetition

Examples

1. In a race 70 runners took part. In how many different orders can they finish? (We assume that everybody completes the race and there are no equal finishes.)
2. For breakfast we can eat
   2 different sandwiches in $2! = 2 \cdot 1 = 2$ different orders.
   3 different sandwiches in $3! = 3 \cdot 2 \cdot 1 = 6$ different orders.
   4 different sandwiches in $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ different orders.
3. A group of 200 students can sign the attendance sheet in $200! = 200 \cdot 199 \cdot 198 \cdot \ldots \cdot 2 \cdot 1 \approx 7,89 \cdot 10^{374}$ different orders.

Permutations with repetition

Example: In an exam 5 students took part and 2 grade 4's and 3 grade 5's were awarded. In how many different orders can we list the results on a paper if we do not distinguish between identical grades (i.e. we do not care about which student each grade belongs to)?

Solution: If we take into account which student obtained each grade (for example, we indicate each student's name next to his/her grade) then there are $(2+3)! = 5!$ possible orders. If we now disregard which student obtained each grade, then in the previous calculation we counted each possible order multiple times:

5	5	5	5	5	5	5	5	5	5	5	5	
5	5	5	5	5	5	5	5	5	5	5	5	
5	5	5	5	5	5	5	5	5	5	5	5	...
4	4	4	4	4	4	4	4	4	4	4	4	
4	4	4	4	4	4	4	4	4	4	4	4	

The grade 5's can be permuted among themselves in $3! = 6$ different ways. Similarly, the grade 4's can be permuted $2! = 2$ different ways amog themselves. Therefore each order of the grades were counted $3!2!$ times. Hence the number of different orders is: $\dfrac{5!}{2! \cdot 3!} = \dfrac{120}{2 \cdot 6} = 10$.

Permutations with repetition

Definition (permutations with repetition)

Let $a_1, a_2, \ldots, a_m$ be m different objects and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. A sequence of length $n = k_1 + k_2 + \ldots + k_m$ containing each a_i exactly k_i times ($1 \leq i \leq m$) is called a permutation with repetition of the objects $a_1, a_2, \ldots, a_m$ with repetition numbers $k_1, k_2, \ldots, k_m$.

Theorem (Number of permutations with repetition)

The number of permutations with repetition of m different objects with repetition numbers $k_1, k_2, \ldots, k_m$ is:

$$\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!},$$

where $n = k_1 + k_2 + \ldots + k_m$.

Permutations with repetition

Theorem (Number of permutations with repetition)

The number of permutations with repetition of m different objects with repetition numbers $k_1, k_2, \ldots, k_m$ is:

$$\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!},$$

where $n = k_1 + k_2 + \ldots + k_m$.

Proof

If we distinguish between all elements, then there are $n! = (k_1 + k_2 + \ldots + k_m)!$ possible sequences of these n different elements.
However, we do not want to distinguish between elements of the same kind, but for each i we are only interested in the set of positions occupied by the elements of the i^{th} kind. If we fix the k_i positions for the elements of the i^{th} kind, we can permute these elements in these positions in $k_i!$ ways. Hence, in $n!$, each sequence has been counted $k_1! \cdot k_2! \cdot \ldots \cdot k_m!$ times. Therefore the number of permutations with repetition is $\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$. $\square$

Variations without repetition (partial permutations)

Example

In a horse race there are 30 runners. How many different outcomes are possible for the first five places?

Solution

The winner can be chosen in 30 ways, then we can choose the horse for the 2^{nd} place in 29 different ways, ..., we can choose the horse for the 5^{th} place in 26 different ways.
Hence there are $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$ possible outcomes for the first five places.

Definition (variation (or partial permutation))

Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by elements of A containing each element of A *at most once*, is called a k-variation (without repetition) of A.

Variations without repetition (partial permutations)

Theorem (Number of variations without repetition)

Let $k \in \mathbb{N}^+$. The number of k-variations without repetition of an n-element set is

$$V_n^k = P_n^k = P(n,k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n!/(n-k)!$$

if $k \leq n$ and is 0 otherwise.

Proof

Let $k \leq n$. The first element of the sequence can be chosen in n different ways from set A. After this, the second element can be chosen in $n-1$ different ways (the first element of the sequence cannot be chosen again), then the third element can be chosen in $n-2$ different ways... the k^{th} element can be chosen in $n-k+1$ different ways. Therefore there are $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n!/(n-k)!$ k-variations in total. If $k > n$ then A clearly does not have a k-variation without repetition. $\square$

Variations with repetition

Example

How many different 2-digit numbers can be formed using the digits 1, 2, 3, if not all digits need to be used and the repetition of the digits is allowed?

Solution

The first digit of the number can be chosen in 3 different ways:

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

The second digit of the number can be chosen again in 3 different ways:

$$\begin{array}{ccc} 11 & 21 & 31 \\ 12 & 22 & 32 \\ 13 & 23 & 33 \end{array}$$

The total number of possibilities:

$$3 \cdot 3 = 9$$

Variations with repetition

Definition (variation with repetition)

Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by elements of A (any element may occur more than once), is called a k-variation with repetition of A.

Theorem (Number of variations with repetition)

Let $n, k \in \mathbb{N}$. The number of k-variations with repetition of an n-element set is: n^k.

Proof

The first element of the sequence can be chosen in n different ways, then the second element can be chosen also in n different ways . . . . Therefore the sequence of length k can be chosen in n^k different ways. $\square$

Variations with repetition

Examples

1. How many different 12-digit numbers can be formed using the digits $1 - 9$, if not all the digits have to be used and the repetition of digits is allowed?

   Each of the 12 digits can be chosen in 9 different ways (independently of each other), hence there are 9^{12} such numbers.

2. How many subsets does an n-element set have?

   Let $A = \{a_1, a_2, \ldots, a_n\}$. To each subset S of A we can assign a $0 - 1$ sequence of length n: for each $1 \leq i \leq n$ let the i^{th} element of the sequence be 1 if S contains a_i and 0 otherwise.
   $\varnothing \leftrightarrow (0, 0, \ldots, 0)$, $\{a_1, a_3\} \leftrightarrow (1, 0, 1, 0, \ldots, 0)$, ...,
   $A \leftrightarrow (1, 1, \ldots, 1)$
   How many $0 - 1$ sequences of length n are there? 2^n.

Combinations without repetition

Definition (combination without repetition)

Let $k \in \mathbb{N}$. A k-element subset of a set A is called a k-combination (without repetition) of A.

Definition (binomial coefficients)

For every $n, k \in \mathbb{N}$, $k \leq n$ the binomial coefficient 'n choose k' is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem (Number of combinations without repetition)

Let $n, k \in \mathbb{N}$. The number of k-combinations of an n-element set is:

$$C_n^k = C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

if $k \leq n$ and is 0 otherwise.

Theorem (Number of combinations without repetition)

Let $n, k \in \mathbb{N}$. The number of k-combinations of an n-element set is:

$$C_n^k = C(n,k) = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

if $k \leq n$ and is 0 otherwise.

Proof

Let A be an n-element set. Suppose $k \leq n$. First choose k elements from among the n elements by taking into account the order. This way we obtain k-variations of A, and the number of these is $\frac{n!}{(n-k)!}$. If we now disregard the order of the elements then each subset of k elements has been counted $k!$ times, because this is how many ways we can arrange k elements into order. Hence, by dividing by $k!$ we obtain that the number of k-element subsets is $\frac{n!}{k! \cdot (n-k)!}$.
If $k > n$ then an n-element set clearly does not have any k-element subsets.

$\square$

Combinations

Examples

1. In how many different ways can you complete a lottery ticket (we need to select 5 numbers from among the numbers $1-90$)?

$$\binom{90}{5} = \frac{90!}{5! \cdot 85!} = \frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 43\,949\,268$$

2. How many $0-1$ sequences of length 12 exist containing exactly seven 1's?

$$\binom{12}{7} = \frac{12!}{7!5!}$$

Combinations with repetition

Definition (combination with repetition)

Let $k \in \mathbb{N}$. A k-combination with repetition (or k-multiset) from a set A is a selection of k (not necessarily distinct) elements from A, where repetition is allowed and the order does not matter.

Comment: In a combination with repetition what matters only is *how many times* each element has been chosen.

Example

In a post office 4 different types of postcards are sold. In how many different ways can we buy 12 postcards?

$$\binom{12+4-1}{12} = \binom{15}{12} = \frac{15!}{12!3!}$$

Combinations with repetition

Theorem (Number of combinations with repetition)

Let $n \in \mathbb{N}$, $k \in \mathbb{N}^+$. The number of k-combinations with repetition of an n-element set is:

$$\binom{n+k-1}{k}.$$

Proof

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then each k-combination with repetition of A can be represented by a $0-1$-sequence:

$$\underbrace{1, 1, \ldots, 1}_{\substack{\text{number of } a_1\text{'s} \\ \text{chosen}}}, 0, \underbrace{1, 1, \ldots, 1}_{\substack{\text{number of } a_2\text{'s} \\ \text{chosen}}}, 0, \ldots, 0, \underbrace{1, 1, \ldots, 1}_{\substack{\text{number of } a_n\text{'s} \\ \text{chosen}}}.$$

This sequence contains k 1's (number of elements chosen) and $(n-1)$ 0's (number of separators). Hence the length of the sequence is $n-1+k$. There are $\binom{n+k-1}{k}$ such sequences in total, because this is how many ways we can choose the k positions from among $n+k-1$ positions, for the 1's. Therefore the number of k-combinations with repetition of A is $\binom{n+k-1}{k}$. $\square$

Combinations with repetition

Examples

1. In a cake shop they sell 5 different types of cakes. We would like to buy 8 cakes. In how many different ways can we do this?

   Here $n = 5$ and $k = 8$:

$$\binom{5+8-1}{8} = \binom{12}{8} = \frac{12!}{8! \cdot 4!} = 495$$

2. In how many different ways can we distribute 11 identical candies among 5 children?

   For each candy we choose one from among the 5 children who to give the candy to. We choose from among the 5 children 11 times: the order in which we choose the children does not matter and any child can be chosen more than once (what matters only is *how many times* each child has been chosen). Combination with repetition where $n = 5$ and $k = 11$:

$$\binom{11+5-1}{11} = \binom{15}{11} = \frac{15!}{11! \cdot 4!}$$

Summary: six basic arrangement types

Permutations without repetition: $n!$, a sequence of n distinct elements, containing each element exactly once (order matters, each element occurs exactly once).

Permutations with repetition: $\frac{(k_1 + k_2 + \ldots + k_m)!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$, a sequence of length $n = k_1 + k_2 + \ldots + k_m$, containing the element of kind i exactly k_i times ($1 \leq i \leq m$) (order matters, an element can occur more than once: the i^{th} kind of element occurs exactly k_i times).

Variations without repetition: $n!/(n-k)!$, a sequence of length k chosen from n different elements, containing each element at most once (order matters, an element can occur at most once).

Variations with repetition: n^k, a sequence of length k chosen from n different elements (order matters, an element can occur more than once).

Combinations without repetition: $\binom{n}{k}$, k-element subset of an n-element set (order does not matter, an element can occur at most once).

Combinations with repetition: $\binom{n+k-1}{k}$, choosing k times from among n elements disregarding the order in which the elements were chosen and allowing for choosing an element more than once (order does not matter, an element can occur more than once).

Binomial theorem

Theorem (Binomial theorem)

For any $x, y \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

$$(x+y)^n = (x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)$$

By the distributive law, the expansion of the above product will be the sum of all those products that can be obtained by choosing x *or* y *from each pair of brackets and multiplying them together. Hence the expansion will be a sum of terms of the form* $x^k y^{n-k}$ $(0 \leq k \leq n)$*. For a given* k *the term* $x^k y^{n-k}$ *occurs in the sum as many times as many different ways we can choose* x *from exactly* k *of the* n *pairs of brackets. From among the* n *pairs of brackets those* k *brackets from which we pick* x *can be chosen in* $\binom{n}{k}$ *different ways. Therefore for every* $0 \leq k \leq n$ *the coefficient of* $x^k y^{n-k}$ *in the expansion is* $\binom{n}{k}$*.* $\square$

Binomial coefficients

Theorem (Some properties of the binomial coefficients)

For every $n, k \in \mathbb{N}$, $k \leq n$*:*

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ *if* $k \geq 1$.

For every $n \in \mathbb{N}$*:*

3. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.
4. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0$.

Binomial coefficients

Theorem (Some properties of the binomial coefficients)

For every $n, k \in \mathbb{N}$, $k \leq n$ *we have:*

1. $\binom{n}{k} = \binom{n}{n-k}$.
2. $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ *if* $k \geq 1$.

Proof

1. *The number of all* $0-1$ *sequences of length* n *containing exactly* k *number of* 1*-s is* $\binom{n}{k}$*, because this is how many ways we can chose the* k *positions for the* 1*-s from among the* n *positions. Since these are exactly the* $0-1$ *sequences of length* n *containing exactly* $n-k$ *number of* 0*-s, their number is* $\binom{n}{n-k}$*, because this is how many ways we can chose the* $n-k$ *positions for the* 0*-s from among the* n *posititions. Therefore* $\binom{n}{k} = \binom{n}{n-k}$.
2. *The number of* $0-1$ *sequences of length* $n+1$ *which contain exactly* k *number of* 1*-s is equal to* $\binom{n+1}{k}$*. Among the* $0-1$ *sequences of length* $n+1$ *containing exactly* k *number of* 1*-s there are* $\binom{n}{k-1}$ *sequences starting with* 1 *and there are* $\binom{n}{k}$ *sequences starting with* 0*. Hence* $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

□

Binomial coefficients

Theorem (Some properties of the binomial coefficients)

For every $n \in \mathbb{N}$ *we have:*

3. $$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$
4. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$

Proof

3. *By our earlier theorem, for every* $0 \leq k \leq n$, *the binomial coefficient* $\binom{n}{k}$ *equals the number of* k*-element subsets of an* n*-element set. Adding up for all* $0 \leq k \leq n$, *the number of* k*-element subsets of an* n*-element set we obtain the number of all subsets of an* n*-element set, which – as shown earlier – is equal to* 2^n.
4. *By the Binomial theorem:* $(-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. *Since* $(-1+1)^n = 0^n = 0$, *we have* $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Pascal's triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}: \ \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

n	$\binom{n}{k}$	$(x+y)^n$
0	1	1
1	1 1	$x+y$
2	1 2 1	$x^2+2xy+y^2$
3	1 3 3 1	$x^3+3x^2y+3xy^2+y^3$
4	1 4 6 4 1	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$
5	1 5 10 10 5 1	$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

Multinomial theorem

Example

Expand the following:

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz.$ $\quad (x+y+z)^3 = \ldots$

Theorem (Multinomial theorem)

For any r and $n \in \mathbb{N}$ we have

$$(x_1+x_2+\ldots+x_r)^n = \sum_{\substack{i_1+i_2+\ldots+i_r=n \\ i_1,i_2,\ldots,i_r\in\mathbb{N}}} \frac{n!}{i_1!\cdot i_2!\cdot\ldots\cdot i_r!} x_1^{i_1}\cdot x_2^{i_2}\cdot\ldots\cdot x_r^{i_r}.$$

Proof

$(x_1+x_2+\ldots+x_r)^n = (x_1+x_2+\ldots+x_r)(x_1+x_2+\ldots+x_r)\cdots(x_1+x_2+\ldots+x_r).$

The coefficient of the term $x_1^{i_1}x_2^{i_2}\ldots x_r^{i_r}$ is equal to:

$$\binom{n}{i_1}\binom{n-i_1}{i_2}\binom{n-i_1-i_2}{i_3}\cdots\binom{n-i_1-i_2-\ldots-i_{r-1}}{i_r} =$$

$$\frac{n!}{i_1!(n-i_1)!}\,\frac{(n-i_1)!}{i_2!(n-i_1-i_2)!}\cdots\frac{(n-i_1-i_2-\ldots-i_{r-1})!}{i_r!(n-i_1-\ldots-i_{r-1}-i_r)!} = \frac{n!}{i_1!\cdot i_2!\cdots i_r!} \qquad \square$$

Multinomial theorem

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{\substack{i_1+i_2+\ldots+i_r=n \\ i_1,i_2,\ldots,i_r \in \mathbb{N}}} \frac{n!}{i_1!\,i_2!\cdots i_r!} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$$

$$(x + y + z)^3 = \ldots$$

i_1	i_2	i_3	$\frac{3!}{i_1!i_2!i_3!}$	$(x+y+z)^3 =$
3	0	0	$\frac{3!}{3!0!0!} = 1$	x^3
2	1	0	$\frac{3!}{2!1!0!} = 3$	$+3x^2y$
2	0	1	$\frac{3!}{2!0!1!} = 3$	$+3x^2z$
1	2	0	$\frac{3!}{1!2!0!} = 3$	$+3xy^2$
1	1	1	$\frac{3!}{1!1!1!} = 6$	$+6xyz$
1	0	2	$\frac{3!}{1!0!2!} = 3$	$+3xz^2$
0	3	0	$\frac{3!}{0!3!0!} = 1$	$+y^3$
0	2	1	$\frac{3!}{0!2!1!} = 3$	$+3y^2z$
0	1	2	$\frac{3!}{0!1!2!} = 3$	$+3yz^2$
0	0	3	$\frac{3!}{0!0!3!} = 1$	$+z^3$

Pigeonhole principle

Pigeonhole principle

If $n+1$ items are put into n containers, then there must be a container that contains at least two items.

Examples

1. In any group of eight people there must be at least two who were born on the same day of the week.
2. If we choose any five (different) numbers from the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, there will always be two numbers among them which add up to 9.

   Consider the sets $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$. By the Pigeonhole principle, among the five numbers chosen from A there will be two numbers which belong to the same set, hence their sum is 9.

Inclusion-exclusion principle

Example

How many positive integers less than 1000 exist which are not divisible by 2, nor by 3, nor by 5?

First consider the following question: How many positive integers less than 1000 exist which are divisible by 2, by 3 or by 5?

$A_1 = \{1 \le n \le 999 : 2|n\} \to |A_1| = \left\lfloor \frac{999}{2} \right\rfloor$;
$A_2 = \{1 \le n \le 999 : 3|n\} \to |A_2| = \left\lfloor \frac{999}{3} \right\rfloor$;
$A_3 = \{1 \le n \le 999 : 5|n\} \to |A_3| = \left\lfloor \frac{999}{5} \right\rfloor$.

Similarly, $|A_1 \cap A_2| = \left\lfloor \frac{999}{2\cdot 3} \right\rfloor$, $|A_1 \cap A_3| = \left\lfloor \frac{999}{2\cdot 5} \right\rfloor$, $|A_2 \cap A_3| = \left\lfloor \frac{999}{3\cdot 5} \right\rfloor$, $|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor$.

The number of positive integers less than 1000 divisible by 2 or by 3 or by 5:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| =$$

$$= \left\lfloor \frac{999}{2} \right\rfloor + \left\lfloor \frac{999}{3} \right\rfloor + \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{999}{2\cdot 3} \right\rfloor - \left\lfloor \frac{999}{2\cdot 5} \right\rfloor - \left\lfloor \frac{999}{3\cdot 5} \right\rfloor + \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor.$$

Inclusion-exclusion principle

Example continued

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| =$$

$$= \left\lfloor \frac{999}{2} \right\rfloor + \left\lfloor \frac{999}{3} \right\rfloor + \left\lfloor \frac{999}{5} \right\rfloor - \left\lfloor \frac{999}{2\cdot 3} \right\rfloor - \left\lfloor \frac{999}{2\cdot 5} \right\rfloor - \left\lfloor \frac{999}{3\cdot 5} \right\rfloor + \left\lfloor \frac{999}{2\cdot 3\cdot 5} \right\rfloor =$$

$$= 499 + 333 + 199 - 166 - 99 - 66 + 33 = 733.$$

The number of positive integers less than 1000, not divisible by 2, nor by 3, nor by 5: $999 - 733 = 266$

Inclusion-exclusion principle

Theorem (Inclusion-exclusion principle)

Let A_1, A_2, ..., A_n be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i_1<i_2\le n} |A_{i_1}\cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le n} |A_{i_1}\cap A_{i_2}\cap A_{i_3}| - \dots$$

$$+(-1)^{n+1}|A_1\cap A_2\cap\ldots\cap A_n|$$

For every $1\le r\le n$ introduce the notation:

$$S_r = \sum_{1\le i_1<i_2<\ldots<i_r\le n} |A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_r}|$$

Then the Inclusion-exclusion principle can be written in the following, simpler form:

$$\left|\bigcup_{i=1}^{n} A_i\right| = S_1 - S_2 + \cdots + (-1)^{n+1} S_n$$

Inclusion-exclusion principle

For every $1 \leq r \leq n$ introducing the notation:

$$S_r = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|$$

the Inclusion-exclusion principle can be written in the following form:

$$\left| \bigcup_{i=1}^{n} A_i \right| = S_1 - S_2 + \cdots + (-1)^{n+1} S_n$$

Proof

We show that every element of $\bigcup_{i=1}^{n} A_i$ was counted exactly once in the expression $S_1 - S_2 + \ldots + (-1)^{n+1} S_n$. Let $x \in \bigcup_{i=1}^{n} A_i$ be arbitrary. Denote by t the number of those sets among $A_1, \ldots, A_n$ which contain x, and the sets containing x by $A_{j_1}, \ldots, A_{j_t}$. Note that for any $1 \leq r \leq n$, element x is contained in the intersection of r sets if and only if all the r sets contain x, that is if all the r sets are among $A_{j_1}, \ldots, A_{j_t}$. Hence for every $1 \leq r \leq n$, we counted x in S_r as many times as many different ways r sets can chosen from among the sets $A_{j_1}, \ldots, A_{j_t}$. Hence, in S_r element x was counted 0 times if $r > t$ and $\binom{t}{r}$ times if $r \leq t$. Therefore in $S_1 - S_2 + \ldots + (-1)^{n+1} S_n$ element x was counted exactly $\binom{t}{1} - \binom{t}{2} + \ldots + (-1)^{t+1} \binom{t}{t}$ times.

Inclusion-exclusion principle

Proof (Proof continued)

Finding the value of $\binom{t}{1} - \binom{t}{2} + \ldots + (-1)^{t+1}\binom{t}{t}$*:*

By Property 4 of the binomial coefficients:

$$\binom{t}{0} - \binom{t}{1} + \binom{t}{2} - \ldots + (-1)^t \binom{t}{t} = 0.$$

By rearranging the above equation:

$$\binom{t}{1} - \binom{t}{2} + \ldots + (-1)^{t+1}\binom{t}{t} = \binom{t}{0} = 1.$$

Hence x *was counted exactly once in the expression*
$S_1 - S_2 + \ldots + (-1)^{n+1} S_n$.