Problem set 6: Geometry of complex numbers and complex roots of unity

Summary of theory

Geometry of complex numbers

Geometric meaning of functions of complex numbers:

Expression	**Geometric meaning** (z and w are represented as points in the complex plain)
$Re(z)$	coordinate of z on the real axis („x coordinate”)
$Im(z)$	coordinate of z on the imaginary axis („y coordinate”)
$\|z\|$	distance of z from the origin
$arg(z)$	angle from the positive real axis to the position vector representing z, in the interval $[0, 2\pi)$
$\overline{z}$	image of z under reflection in the real axis
$-z$	image of z under relflection in the origin
$\|z-w\|$	distance between z and w
$\frac{z+w}{2}$	midpoint of the line segment with endpoints z and w
$\alpha z + (1-\alpha)w$ ($0 \le \alpha \le 1$ is a real number)	point dividing the line segment between z and w in the ratio $(1-\alpha) : \alpha$ ($1-\alpha$ is the part on the z side)

Geometric transformations of the complex plane determined by operations of complex numbers:

Function $\mathbb{C} \to \mathbb{C}$	**Transformation of the complex plane/$\mathbb{R}^2$**
$z \mapsto z + w$	translation by vector $\binom{Re(w)}{Im(w)}$
$z \mapsto zw$	enlargement by a factor of $\|w\|$ with a rotation by angle $arg(w)$ around the origin
$z \mapsto -z$	reflection in the origin
$z \mapsto \frac{z}{w}$ ($w \neq 0$)	enlargement by a factor of $\frac{1}{\|w\|}$ with a rotation by angle $-arg(w)$ around the origin
$z \mapsto \overline{z}$	reflection in the real axis

Roots of unity

n^{th} roots of unity: Let $n \in \mathbb{N}^+$. A complex number ε is an n^{th} *root of unity*, if $\varepsilon^n = 1$.
roots of unity: A complex number ε is a *(complex) root of unity*, if it is an n^{th} root of unity for some $n \in \mathbb{N}^+$.
primitive n^{th} roots of unity: If n is the smallest positive integer such that $\varepsilon^n = 1$ then ε is called a *primitive n^{th} root of unity.*
The polar form of the n^{th} roots of unity: For any $n \in \mathbb{N}^+$ the n^{th} roots of unity are:

$$\varepsilon_k = \varepsilon_k^{(n)} = (\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}) \text{ where } k = 0, 1, \ldots, n-1.$$

Expressing all n^{th} roots of a complex number using one n^{th} root and the n^{th} roots of unity: Let $z \in \mathbb{C}$ be a nonzero complex number, $n \in \mathbb{N}^+$ and $w \in \mathbb{C}$ be such that $w^n = z$. Then the n^{th} roots of z can be expressed in the following form:

$$w_k = w\varepsilon_k^{(n)} \text{ where } k = 0, 1, \ldots, n-1.$$

Questions

Geometry of complex numbers

Question 1: In each of the following examples describe the geometric transformation of the Gaussian plane determined by the given function.
(a) $z \mapsto 3z + 2$
(b) $z \mapsto (1 + i)z$
(c) $z \mapsto 1/\overline{z}$

Question 2: Find the vector obtained by rotating the vector $\begin{bmatrix} 2 \\ -2\sqrt{3} \end{bmatrix} \in \mathbb{R}^2$ in the plane by each of the following angles: (a) $45°$ (b) $30°$ (c) $-60°$.

Question 3: Consider the sets

$$A = \{z \in \mathbb{C} \mid \operatorname{Re} z > 1\}$$
$$B = \{z \in \mathbb{C} \mid \operatorname{Im} z < 2\}$$
$$C = \{z \in \mathbb{C} \mid |z - 2| = 3\}$$
$$D = \{z \in \mathbb{C} \mid z^2 - (3 + 2i)z + (5 + 5i) = 0\}$$

Represent each of the following sets in the Gaussian plane:

(a) A	(b) B	(c) C	(d) D	(e) $A \cap B$	(f) $A \cup B$
(g) $A \cap C$	(h) $B \cup C$	(i) $A \setminus B$	(j) $A \triangle B$	(k) $A \cap D$	(l) $C \setminus \overline{B}$

Question 4: Represent each of the following sets in the Gaussian plane:
(a) $\{z \in \mathbb{C} \mid |z - i + 2| = 10\}$
(b) $\{z \in \mathbb{C} \mid \operatorname{Re} z = \operatorname{Im} z\}$
(c) $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq \operatorname{Im} z\}$
(d) $\{z \in \mathbb{C} \mid |z - 2| \leq |z + 3|\}$
(e) $\{z \in \mathbb{C} \mid 2 < |z + i - 2| \leq 4\}$

Question 5: In the Gaussian plane the center of a square is at point $K = 1 + 2i$ and one of the vertices of this square is point $A = 5 + 4i$. Determine the complex numbers represented by the other three vertices of this square.

Question 6: Let $z \neq w$ be two complex numbers. Find the complex number represented by the midpoint of the line segment between z and w. Consider the two equilateral triangles having both z and w among their vertices. Find the complex number represented by the third vertex of each of these two equilateral triangles. Find the complex number represented by the centroid (or median point) of each triangle.

Complex roots of unity

Question 7: For each number z below, decide if z is a complex root of unity. If z is a complex root of unity then:

(1) find the order of z;

(2) find all those values $n \in \mathbb{N}^+$ for which z is an n^{th} root of unity and

(3) find those values $n \in \mathbb{N}^+$ for which z is a primitive n^{th} root of unity.

(a) 1 (b) -1 (c) i

(d) $1+i$ (e) $\dfrac{1+i}{\sqrt{2}}$ (f) $\dfrac{1+\sqrt{3}i}{2}$

(g) $\dfrac{-1+\sqrt{3}i}{2}$ (h) $\dfrac{-1+\sqrt{3}i}{2}$ (i) $\cos\left(\sqrt{2}\pi\right) + i\sin\left(\sqrt{2}\pi\right)$

(j) $\cos\left(\dfrac{\pi}{361}\right) + i\sin\left(\dfrac{\pi}{361}\right)$

Question 8: Show that if $\varepsilon^4 = i$ then $4 \mid o(\varepsilon)$.

Further qestions

Question 9: Suppose $o(\varepsilon) = 128$. What is $o(i \cdot \varepsilon)$? Justify your answer.

Question 10:

(a) One of the fourth roots of $z = -1 - \sqrt{3}i$ is $w_0 = \dfrac{\sqrt[4]{2}}{2}(\sqrt{3} - i)$. Using a primitive fourth root of unity, find all fourth roots of unity. With the help of these, calculate all fourth roots of z.

(b) One of the sixth roots of $z = -i$ is $w_0 = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$. Using a primitive sixth root of unity, find all sixth roots of unity. With the help of these, calculate all sixth roots of z.

(Please do not use the formula for calculating the n^{th} roots of a complex number in this question.)