Problem set 3: Properties of homogeneous binary relations, equivalence relations and partitions

Summary of theory

Properties of homogeneous binary relations

homogeneous binary relation: A binary relation $R \subseteq X \times X$ is called a *homogeneous binary relation (on set X)* or a binary relation *on set X*.

A binary relation $R \subseteq X \times X$ on set X is:

- *transitive,* if $\forall x, y, z \in X : (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$;
- *reflexive,* if $\forall x \in X : (x, x) \in R$;
- *irreflexive,* if $\forall x \in X : (x, x) \notin R$;
- *symmetric,* if $\forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \in R$;
- *anti-symmetric,* if $\forall x, y \in X : (x, y) \in R \wedge (y, x) \in R \Rightarrow x = y$;
- *strictly anti-symmetric,* if $\forall x, y \in X : (x, y) \in R \Rightarrow (y, x) \notin R$;
- *dichotomous,* if: $\forall x, y \in X$ at least one of $(x, y) \in R$ and $(y, x) \in R$ holds;
- *trichotomous,* if: $\forall x, y \in X$ exactly one of $(x, y) \in R$, $(y, x) \in R$ and $x = y$ holds.

Equivalence relations, equivalence classes, partitions

equivalence relation: A relation $\sim \subseteq X \times X$ is an *equivalence relation*, if it is reflexive, symmetric and transitive.

equivalence class: Let $\sim \subseteq X \times X$ be an equivalence relation on set X. The *equivalence class* of any $x \in X$: $\overline{x} = [x] = \{y \in X \mid y \sim x\}$ (i.e. the set of those elements in X which are $\sim$-related to x).

partition: Let $X \neq \emptyset$ be a set. A system of sets $\mathcal{P}$ is called a *partition* of X if: (1) all elements of $\mathcal{P}$ are nonempty subsets of X, (2) $\mathcal{P}$ is a pairwise disjoint system and (3) $\bigcup \mathcal{P} = X$. The elements of $\mathcal{P}$ are called the *blocks* or *cells* of the partition.

partition determined by an equivalence relation: Let $\sim \subseteq X \times X$ be an equivalence relation on a set $X \neq \emptyset$. The *partition determined by* $\sim$ (or the *quotient set of X by* $\sim$) is defined as: $X/\sim := \{[x] \mid x \in X\}$ (i.e. as the set of all equivalence classes of $\sim$).

equivalence relation determined by a partition: Let $\mathcal{P}$ be a partition of a set $X \neq \emptyset$. The *equivalence relation determined by* $\mathcal{P}$ is defined as: $\sim := \{(x, y) \mid x$ and y are contained by the same block of $\mathcal{P}\}$.

Questions

Properties of homogeneous binary relations

Question 1: Let $X = \{1, 2, 3\}$. In each of the following examples below decide if the relation ρ on X is reflexive, symmetric, anti-symmetric and/or transitive.

(a) $\rho = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
(b) $\rho = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$
(c) $\rho = \{(1, 2), (1, 3), (2, 1), (3, 1)\}$
(d) $\rho = \{(1, 2), (2, 3), (3, 1)\}$

(e) $\rho = \{(1,2)\}$
(f) $\rho = \{(1,2),(2,1),(2,3),(3,2)\}$
(g) $\rho = \{(1,1),(2,2),(2,3),(3,3)\}$
(h) $\rho = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

Question 2:
(a) Can a relation be both symmetric and anti-symmetric at the same time? Can a relation be both reflexive and irreflexive? Justify your answers.
(b) Prove that if a relation is both symmetric and anti-symmetric then it is also transitive.
(c) Prove that if a relation that is not the empty set is both irreflexive and symmetric, then it is not transitive.

Question 3: In each of the following examples decide if the given relation is reflexive, irreflexive, symmetric, anti-symmetric and/or transitive, and find the domain and the range of the relation.
(a) $R = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \cdot b \text{ is odd}\}$
(b) $S = \{(a,b) \in B \times B \mid$ the surname of a is shorter than the surname of $b\}$ where B is the set of all Discrete maths I students at ELTE.
(c) $T_X = \{(A,B) \in P(X) \times P(X) \mid A \cap B \neq \emptyset\}$ where X is a given set.
(d) $U = \{(x,y) \in K \times K| \mid x$ touches y from inside$\}$, where K is the set of all circles in a given plane.

Question 4: In each question below give an example for a relation on the set $\{1,2,3,4\}$ satisfying simultaneously all the properties listed:

(a) reflexive and not irreflexive;
(b) anti-symmetric and not symmetric;
(c) symmetric and not anti-symmetric;
(d) both symmetric and anti-symmetric;
(e) neither symmetric nor anti-symmetric;
(f) both reflexive and trichotomous;
(g) not reflexive, not transitive, not symmetric, not anti-symmetric and not trichotomous.

Equivalence relations and partitions

Question 5: For part (a) and part (b) below, solve questions (1) and (2).
(a) $\rho = \{(1,1),(1,5),(2,2),(3,3),(3,4),(4,3),(4,4),(5,1),(5,5)\}$ on set $A = \{1,2,3,4,5\}$.
(b) $\rho = \{(1,1),(1,5),(1,6),(1,8),(2,2),(2,4),(3,3),(3,7),(4,2),(4,4),(5,1),(5,5),(5,6),$ $(5,8),(6,1),(6,5),(6,6),(6,8),(7,3),(7,7),(8,1),(8,5),(8,6),(8,8)\}$ on set $A = \{1,2,3,4,5,6,7,8\}$.

(1) Prove that ρ is an equivalence relation on A.
(2) Write down the partition of A determined by the equivalence relation ρ (in other words: Find the quotient set A/ρ).

Question 6: For each partition of $\{a,b,c,d,e,f\}$ below, find the equivalence relation determined by the partition:

(a) $\{\{a,b,f\},\{c\},\{d,e\}\}$
(b) $\{\{a\},\{b\},\{c\},\{d\},\{e,f\}\}$

Question 7: In each of the following examples prove that R is an equivalence relation (on the set which R is defined on in the example), and find the equivalence classes of R.
(a) $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + n \text{ is even}\}$
(b) $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m - n \text{ is divisible by } 3\}$
(c) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x^2 + y^2 \text{ is even}\}$
(d) $R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a - b \text{ is rational}\}$
(e) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 + y_1 = x_2 + y_2\}$
(f) $R = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_1 \cdot y_1 = x_2 \cdot y_2\}$

Further questions

Question 8: Let $R, S \subseteq A \times A$ be symmetric relations. Prove that $R \circ S$ is symmetric if and only if $R \circ S = S \circ R$.

Question 9: Let X be a set. Prove that the intersection of two reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric,transitive) relations on X is also a(n) reflexive (irreflexive, symmetric, anti-symmetric, strictly anti-symmetric,transitive) relation. Is the statement also true for the union (instead of the intersection) of the two relations?

Question 10: How many different

(a) reflexive; (c) symmetric; (e) strictly anti-symmetric; (g) trichotomous
(b) irreflexive; (d) anti-symmetric; (f) dichotomous

binary relations exist on an n-element set? Prove your answers. (Note: There is no general formula known for the number of transitive relations on an n-element set!)

Question 11: Write a program the input of which is a finite set A and a binary relation ρ on A and the program decides about ρ if it is

(a) reflexive; (c) symmetric; (e) strictly anti-symmetric; (g) trichotomous
(b) irreflexive; (d) anti-symmetric; (f) dichotomous (h) transitive.

Question 12: Write a program the input of which is a finite set A and a binary relation ρ on A, which decides about ρ if it is an equivalence relation, and if it is then it finds the partition of A determined by ρ.