Problem set 1: Sets

Summary of theory

Some special sets
empty set: The *empty set* (i.e. the set that has no elements) is denoted by $\emptyset = \{\}$ or $\emptyset$.

system of sets: A set such that all of its elements are also sets is called a *system of sets*.

Subsets, proper subsets
subset: A set A is a *subset* of set B, in notation: $A \subseteq B$, if all elements of A are also elements of set B, that is, if: $\forall x(x \in A \Rightarrow x \in B)$.

proper subset: If A is a subset of B and $A \neq B$, then A is a *proper subset* of B, in notation: $A \subsetneq B$.

Set union and set intersection
set union: The *union* $A \cup B$ of sets A and B is the set which contains all elements of A and B, that is: $A \cup B = \{x \mid x \in A \vee x \in B\}$. In general: Let $\mathcal{A}$ be a system of sets. Then $\bigcup \mathcal{A}$ is the set which contains all elements of the elements of $\mathcal{A}$, that is: $\cup\mathcal{A} = \{x \mid \exists A \in \mathcal{A} : x \in A\}$.

set intersection: The *intersection* $A \cap B$ of sets A and B is the set containing exactly the common elements of A and B, that is: $A \cap B = \{x \mid x \in A \wedge x \in B\}$. In general: Let $\mathcal{A}$ be a system of sets. Then the intersection $\cap\mathcal{A}$ is the set which contains those elements which are elements of all sets in $\mathcal{A}$, that is: $\cap\mathcal{A} = \{x \mid \forall A \in \mathcal{A} : x \in A\}$.

Properties of set union and set intersection: For every set A, B and C:

1. $A \cup \emptyset = A$
2. $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)
3. $A \cup B = B \cup A$ (commutativity)
4. $A \cup A = A$ (idempotence)
5. $A \subseteq B \Leftrightarrow A \cup B = B$
6. $A \cap \emptyset = \emptyset$
7. $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)
8. $A \cap B = B \cap A$ (commutativity)
9. $A \cap A = A$ (idempotence)
10. $A \subseteq B \Leftrightarrow A \cap B = A$

Distributive properties of set union and set intersection: For every set A, B and C:

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Set difference and set complement
set difference: The *difference of* sets A and B is the set $A \setminus B = \{x \in A : x \notin B\}$.

set complement: Let U be the universal set. Then for every set $A \subseteq U$ the *complement of* A is defined as $\overline{A} = A' = U \setminus A$.

Expressing set difference using set intersection and complement: Suppose a universal set is given. Then for every set A and B: $A \setminus B = A \cap \overline{B}$.

Properties of set complement: Let U be the universal set. Then for every set A and B:

1. $\overline{\overline{A}} = A$;
2. $\overline{\emptyset} = U$;
3. $\overline{U} = \emptyset$;
4. $A \cap \overline{A} = \emptyset$;
5. $A \cup \overline{A} = U$;
6. $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$;
7. $\overline{A \cap B} = \overline{A} \cup \overline{B}$;
8. $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Properties 7 and 8 are called *De Morgan's laws (for sets)*.

Symmetric difference of sets
symmetric difference: The *symmetric difference* of sets A and B is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
Alternative expression of the symmetric difference: For every set A and B: $A \triangle B = (A \cup B) \setminus (B \cap A)$.

Cartesian product of sets
Cartesian product: The *Cartesian product* of sets A and B is: $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$.

Power set of a set
power set: The *power set* $\mathcal{P}(A)$ of a set A is the set of all subsets of A, that is $\mathcal{P}(A) = \{S \mid S \subseteq A\}$.

Questions

Elements and subsets of sets and set operations $\cup$, $\cap$, $\setminus$ and set complement

Question 1: Let $U = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set, $A = \{x \in \mathbb{N} \mid 1 \leq x \leq 4\}$, $B = \{0,2,4,8\}$ and $C = \{2,3,5,7\}$.

(a) Write down the following sets explicitly, i.e. by listing all their elements:

(i) $A \cap B$ (ii) $B \cup C$ (iii) $A \setminus C$ (iv) $\overline{C}$

(b) Consider the systems of sets $X = \{A, B, C\}$ and $Y = \{\{0,2,4,6,8\},\{1,3,5,7,9\}\}$.
Find the following sets:

(i) $\cap X$ (ii) $\cup X$ (iii) $X \cup Y$ (iv) $X \cap Y$

(c) Determine the truth value of each of the following statements:

(i) $4 \in B$	(iv) $3 \in A \cap B$	(vii) $A \subseteq \cup Y$	(x) $\{2\} \subseteq A$
(ii) $A \subseteq B$	(v) $\{1,2\} \subseteq A$	(viii) $C \cap \emptyset = \emptyset$	(xi) $2 \in \cup X$
(iii) $\{\emptyset\} \subseteq \cup X$	(vi) $A \in \cup Y$	(ix) $2 \subseteq A$	(xii) $\{2\} \in \cap X$

Question 2: Let $\mathcal{A} = \{\{a,b,c\},\{a,d,e\},\{a,f\}\}$. Find the sets $\cup\mathcal{A}$ and $\cap\mathcal{A}$.

Question 3: Consider the system of sets $X = \{\{1,2,3\},\{2,3,4,5\},\{0,2,3,7\}\}$. Find the following sets:

(a) $\cap X$
(b) $X \cup \{5,6,7,8\}$
(c) $X \cup \{\{3,5,7\},\{1\},\{2\}\}$
(d) $\cup(X \cup \{\{3,5,7\},\{1\},\{2\}\})$
(e) $\cap(X \cup \{\{3,5,7\},\{1\},\{2\}\})$

Question 4: Find the sets A, B and C, given that they satisfy the following:
$A \setminus B = \{1,3,5\}$, $A \cup B \cup C = \{1,2,3,4,5,6\}$, $(A \cap C) \cup (B \cap C) = \emptyset$, $C \setminus B = \{2,4\}$ and $(A \cap B) \setminus C = \{6\}$.

Question 5: Give an example for sets A, B and C that satisfy all the conditions below:

$$A \cap B \neq \emptyset, \quad A \cap C = \emptyset, \quad (A \cap B) \setminus C = \emptyset.$$

Question 6: Let $A = \{a, b, c, d\}$, $B = \{c, d\}$ and $C = \{a, c, e\}$. Show that then $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. Is this statement true for all sets A, B and C?

Proving or disproving set identities

Question 7: Using the properties of logical operations learnt, prove that the following equalities are true for any universal set U and sets $A, B, C \subseteq U$. (Hence, these equalities are identities.)

(a) $A \cup B = B \cup A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

(c) $A \cap B = B \cap A$

(d) $(A \cap B) \cap C = A \cap (B \cap C)$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(h) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(i) $A \cup \overline{A} = U$

(j) $A \cap \overline{A} = \emptyset$

(k) $\overline{\overline{A}} = A$

Question 8: Prove that the following equalities hold for every set A and B:

(a) $(A \setminus B) \cap B = \emptyset$

(b) $\left(A \cup \overline{B}\right) \cap \left(\overline{A} \cup \overline{B}\right) = \overline{B}$

Question 9: Show that the following statements are true for all sets A, B and C:

(a) if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$

(b) if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$

(c) $A \cup (B \cap A) = A$

Question 10: Write the following expression in its simplest possible form:

$$(A \cup (A \cap B) \cup (A \cap B \cap C)) \cap (A \cup B \cup C).$$

Question 11: Prove that the following equalities hold for all universal sets U and sets $A, B, C \subseteq U$. (Hence, these equalities are identities.)

(a) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$

(b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(c) $A \setminus \ (A \setminus (B \setminus C)) = A \cap B \cap \overline{C}$

Question 12: Prove the following identity: $\overline{(\overline{A \cap B} \cup C) \cap \overline{A}} \cup \overline{B} \cup \overline{C} = A \cup \overline{B} \cup \overline{C}$.

Question 13: Decide which of the following statements are true for all sets A, B and C. Prove your answers.

(a) $\overline{A} \cap B = B \setminus A$

(b) $(A \cap B) \setminus C = (A \setminus B) \cap C$

(c) $(A \cup B) \cap (B \setminus A) = (A \cup B) \setminus (A \setminus B)$

(d) $(A \cup B) \setminus A = B$

(e) $(A \cup B) \setminus C = A \cup (B \setminus C)$

Symmetric difference of sets

Question 14: Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:

(a) $A \triangle B$ (b) $A \triangle C$ (c) $(A \triangle B) \triangle C$ (d) $A \triangle (B \triangle C)$

Question 15: Prove the following identities.

(a) $A \triangle \emptyset = A$

(b) $A \triangle A = \emptyset$

(c) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$

(d) $A \triangle (A \triangle B) = B$

Cartesian product of sets

Question 16: Let $A = \{1, 2\}, B = \{a, b, c\}$ and $C = \{2, 3, 4\}$. Find the following sets:

(a) $A \times B$

(b) $B \times A$

(c) $A \times A$

(d) $(A \times A) \times B$

(e) $A \times (A \times B)$

(f) $A \times A \times B$

Question 17: Prove that for every nonempty set A, B, C and D, $A \times B \subseteq C \times D$ holds if and only if $A \subseteq C$ and $B \subseteq D$.

Question 18: Prove that the following is true for all sets A, B and C:

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$

Power set of a set

Question 19: Find the power set $\mathcal{P}(A)$ of A when

(a) $A = \{a, b\}$ (b) $A = \{a, b, c\}$ (c) $A = \{1, 2, 3\}$ (d) $A = \{a, b, c, d\}$

Question 20: Prove that for every sets A and B we have $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, where $\mathcal{P}(A)$ denotes the power set of A. What can we say about the truth value of the statement obtained by replacing $\cap$ by $\cup$?

Further questions

Question 21: Based on Question 15 part c, for every set A, B and C we have $A \triangle (B \triangle C) = (A \triangle B) \triangle C$. Therefore the notation $A \triangle B \triangle C := A \triangle (B \triangle C) = (A \triangle B) \triangle C$ can be introduced. Prove that for every set A, B, C and D we have $A \triangle (B \triangle C \triangle D) = (A \triangle B \triangle C) \triangle D$.

Question 22: Natural numbers can be defined by sets recursively as follows:

- $0 := \emptyset$, $1 := \{\emptyset\}$ and
- $\forall n \in \mathbb{N}, n > 0 : n + 1 := n \cup \{n\}$.

(For example, by the above definition: $2 = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.) Write down the numbers 3 and 4 represented as sets according to the above definition.