Problem set 4: Partial orders, functions

Summary of theory

Partial orders

partial order, partially ordered set: A binary relation $\preceq \subseteq X \times X$ on set X is a *partial oder* if it is reflexive, transitive and anti-symmetric. Then $(X, \preceq)$ is a *partially ordered set.*

Let $(X, \preceq)$ be a partially ordered set. An element $x \in X$ is called:

- *greatest element*, if: $\forall y \in X : y \preceq x$.
- *maximal element*, if: $\nexists y \in X$: $y \neq x \wedge x \preceq y$.
- *least element*, if: $\forall y \in X : x \preceq y$.
- *minimal element*, if: $\nexists y \in X$: $y \neq x \wedge y \preceq x$.

immediate predecessor: Let $(X, \preceq)$ be a partially ordered set. For any $x, y \in X$, $x \neq y$ we say that x is an *immediate predecessor of* y if $\nexists z \in X : (z \neq x \wedge z \neq y \wedge x \preceq z \preceq y)$.

Hasse-diagram: A (finite) partially ordered set $(X, \preceq)$ can be represented on a Hasse-diagram as follows: each element of X is represented by a 'dot'on the diagram. Two dots representing x and y, respectively are connected by a line if and only if x is a predecessor of y or y is a predecessor of x. In this case we place the dot representing the element which is a predecessor of the other one lower on the diagram than the dot representing the other element.

comparable and incomparable elements: Two elements x and y of a partially ordered set $(X, \preceq)$ are said to be *comparable*, if $x \preceq y$ or $y \preceq x$ holds; otherwise x and y are said to be *incomparable.*

(total) order: A partial order $\preceq \subseteq X \times X$ is called a *total order* (or *order* for short), if every pair of elements $x, y \in X$ is comparable, that is, if: $\forall x, y \in X : x \preceq y$ or $y \preceq x$. In oder words: a *(total) order* is a dichotomous partial order. In this case the pair $(X, \preceq)$ is called a *totally ordered set* (or *ordered set* for short).

Functions

function: A relation $f \subseteq X \times Y$ is a *function* if: $\forall x, y, y' : (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$.

Notations:

- If f is a function then $f(x) = y \Leftrightarrow (x, y) \in f$.
- $X \to Y$ denotes the set of all functions $f \subseteq X \times Y$.
- $f \in X \to Y \Leftrightarrow f \subseteq X \times Y$ is a function.
- $f : X \to Y \Leftrightarrow (f \in X \to Y \wedge dmn(f) = X)$.

A function $f : X \to Y$ is

- *injective* if: $\forall x_1, x_2, y : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$;
- *surjective*, if $rng(f) = Y$ and
- *bijective*, if it is both injective and surjective.

Questions

Partial orders

Question 1: Let $A = \{2, 3, 6, 8, 9, 12, 18\} \subseteq \mathbb{N}^+, R \subseteq A \times A$ and $aRb \iff a \mid b$.
(a) Prove that R is a partial order on set A.
(b) Draw the Hasse-diagram of the partial order R.

Question 2:
(a) Prove that the relation $\preceq$ is a partial order on $\mathbb{N}$, where $\preceq$ is defined as follows: $\forall n, m \in \mathbb{N} : n \preceq m \iff \exists k \in \mathbb{N}$ such that $n + k = m$.
(b) Define the binary relation R on $\mathbb{N}\times\mathbb{N}$ as follows: $\forall m_1, m_2, n_1, n_2 \in \mathbb{N} : (m_1, n_1)R(m_2, n_2) \iff m_1 \leq m_2 \wedge n_1 \leq n_2$. Prove that R is a partial order on $\mathbb{N} \times \mathbb{N}$.

Question 3: In each of the following examples decide if relation R is a partial order on the underlying set.
(a) P is the set of all polynomials with real coefficients and $R \subseteq P \times P$, $fRg \iff \deg f \leq \deg g$
(b) $R \subseteq \mathbb{Z} \times \mathbb{Z}, aRb \iff |a| \leq |b|$
(c) V is the set of all those vectors in $\mathbb{R}^2$ which are 10 units in length and $R \subseteq V \times V, xRy \iff$ the angle from the positive real axis to vector x is less than or equal to the angle from the positive real axis to vector y (we assume both of these angles to be in the interval $[0; 2\pi[$)
(d) $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$, $xRy \iff$ the length of vector x is less than or equal to the length of vector y.

Question 4: Decide which of the following relations are total orders on the set $A = \{1, 2, 3, 4\}$.
(a) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
(b) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
(c) $f = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 4)\}$

Functions

Question 5: In each of the following examples decide if the relation f is a function. If f is a function then determine the domain and range of f and decide whether f is surjective, injective and/or bijective.
(a) $A = \{1, 2, 3, 4, 5\}, B = \{10, 11, 12, 13, 14\}, f \subseteq A \times B, f = \{(1, 11), (2, 11), (4, 12), (5, 10)\}$
(b) $A = \{1, 2, 3, 4\}, B = \{a, b, c, d, e, f\}, f \subseteq A \times B, f = \{(1, a), (2, c), (3, e), (3, f), (4, a)\}$
(c) $A = \{1, 2, 3, 4, 5\}, B = \{a, b, c, d, e, f\}, f \subseteq A \times B, f = \{(1, a), (4, e), (5, d)\}$
(d) $A = \{1, 2, 3\}, B = \{1, 3, 5\}, f \subseteq A \times B, f = \{(1, 1), (2, 5), (3, 5)\}$

Question 6: Let $m \in \mathbb{R}^+$ and $A = \{$all isosceles triangles with height of m (from base)$\}$, $B = \mathbb{R}^+$. Define the binary relation $R \subseteq A \times B$ as follows: $aRb, a \in A, b \in B$, if the area of a equals b. Show that R is a function, and examine the properties of f (i.e. decide if f is surjective, injective and/or bijective).

Question 7:
(a) Let $f : \mathbb{R} \to \mathbb{R}, f(x) := 3x - 4$. Prove that function f is bijective, and find the inverse of f.
(b) Let $g : \mathbb{R} \to \mathbb{R}, g(x) := 3 - |x|$. Prove that function g is neither injective, nor surjective.

Question 8: In each of the following examples decide whether f is a function.
(a) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x \mid y$
(b) $f \subseteq \{0,3,5\} \times \{1,2,5\}, xfy \iff xy = 0$
(c) $f \subseteq \{1,2,5\} \times \{0,3,5\}, xfy \iff xy = 0$
(d) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff$ the set of digits contained by the base-10 form of x equals the set of digits in the base-10 form of y.
(e) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff 2x = y$
(f) $f \subseteq \mathbb{Z} \times \mathbb{Z}, xfy \iff x^2 = y^2$
(g) $f \subseteq \mathbb{N} \times \mathbb{N}, xfy \iff x^2 = y^2$
(h) $f \subseteq \mathbb{R} \times \mathbb{R}, xfy \iff x^2 + y^2 = 9$

Question 9: In each of the following examples decide if the given binary relation is a function.
(a) $f_1 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid 7x = y^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(b) $f_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2 + 6y\} \subseteq \mathbb{R} \times \mathbb{R}$
(c) $f_3 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid 7x^2 - 6 = y\} \subseteq \mathbb{R} \times \mathbb{R}$
(d) $f_4 = \{(x,y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid y = |x|\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(e) $f_5 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = (x+4)^2\} \subseteq \mathbb{R} \times \mathbb{R}$
(f) $f_6 = \{(x,y) \in \mathbb{R} \times \mathbb{R}_0^+ \mid 2y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R}_0^+$
(g) $f_7 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 7 \mid x - y\} \subseteq \mathbb{Z} \times \mathbb{Z}$
(h) $f_8 = \{(x,y) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \mid xy = 1\} \subseteq (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$
(i) $f_9 = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(j) $f_{10} = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 3\} \subseteq \mathbb{Z} \times \mathbb{Z}$
(k) $f_{11} = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y(1 - x^2) = x - 1\} \subseteq \mathbb{R} \times \mathbb{R}$
(l) $f_{12} = \{(x,y) \in (\mathbb{R}\setminus\{1,-1\}) \times (\mathbb{R}\setminus\{1,-1\}) \mid y(1-x^2) = x-1\} \subseteq (\mathbb{R}\setminus\{1,-1\}) \times (\mathbb{R}\setminus\{1,-1\})$
About each relation that is a function decide if it is injective, surjective and/or bijective. About each relation that is not a function and is a homogeneous relation, decide if it is reflexive, symmetric and/or transitive.

Further questions

Question 10: Prove that the inverse of a partial order is also a partial order.

Question 11: Prove the following statements:

(a) The greatest (least) element in a partially ordered set (if it exists) is always a maximal (minimal) element. However, a maximal (minimal) element is not necessarily a greatest (least) element.
(b) Every finite partially ordered set contains at least one maximal (minimal) element.
(c) If a partially ordered set contains a greatest (or a least) element then it is unique.
(d) In a totally ordered set an element is a maximal (minimal) element if and only if it is the greatest (least) element.

Question 12: Give an example for a partially ordered set which

(a) does not contain any maximal nor any minimal elements;
(b) contains maximal (minimal) element(s), but no greatest (least) element.
(c) contains more than one maximal (minimal) element.

Question 13: Prove that the inverse of a function f is also a function if and only if f is injective.