Problem set 7: Permutations, variations, combinations

Summary of theory

permutations (without repetition): A *permutation* (without repetition) of a finite set A is a sequence containing each element of A exactly once. (In other words, a possible ordering of the elements of A.)

The number of permutations of a finite set: The number of permutations of an n-element set is $P_n = n!$. (By definition, for every $n \in \mathbb{N}^+$: $n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$, and $0! = 1$.)

permutations with repetition: Let $a_1, a_2, \ldots, a_m$ be m different objects and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. A sequence of length $n = k_1 + k_2 + \ldots + k_m$ containing each a_i exactly k_i times ($1 \le i \le m$) is called a *permutation with repetition* of $a_1, a_2, \ldots, a_m$, *with repetition numbers* $k_1, k_2, \ldots, k_m$.

The number of permutations with repetition: The number of permutations with repetition of m different objects with repetition numbers $k_1, k_2, \ldots, k_m$ is: $\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$, where $n = k_1 + k_2 + \ldots + k_m$.

variations (without repetition) (or partial permutations): Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by some elements of A containing each element of A at most once, is called a *k-variation without repetition* of A.

The number of variations: Let $k^+ \in \mathbb{N}$. The number of k-variations without repetition of an n-element set is: $V_n^k = P_n^k = P(n,k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = n!/(n-k)!$ if $k \le n$ and is 0 if $k > n$.

variations with repetition: Let A be a set and $k \in \mathbb{N}$. A sequence of length k formed by some elements of A (any element may occur more than once), is called a *k-variation with repetition* of A.

The number of variations with repetition: Let $n, k \in \mathbb{N}$. The number of k-variations with repetition of an n-element set is: n^k.

combinations (without repetition): Let $k \in \mathbb{N}$. A k-element subset of a set A is called a *k-combination* of A.

The number of combinations: Let $n, k \in \mathbb{N}$. The number of k-combinations of an n-element set is: $C_n^k = C(n,k) = \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$ if $k \le n$ and is 0 if $k > n$.

combinations with repetition: Let $k \in \mathbb{N}$. A *k-combination with repetition* (or *k-multiset*) from a set A is a selection of k (not necessarily distinct) elements from A, where repetition is allowed and the order does not matter.

The number of combinations with repetition: Let $n, k \in \mathbb{N}$. The number of k-combinations with repetition of an n-element set is: $\binom{n+k-1}{k}$.

Questions

Question 1: In how many ways can you arrange 1, 2, 3 or 5 distinct characters, respectively, into order?

Question 2:
(a) At a literary event 5 different poems are presented. In how many different orders can the poems be read out?
(b) In how many different orders can 6 people be seated along a bench?

(c) 12 students have agreed to have a meeting. In how many different orders can they arrive at the meeting if we assume that they all arrive at different times?
(d) How does the answer to question (b) change if instead of a bench, the 6 people are seated around a round table?

Question 3: In how many different ways can you arrange into order
(a) 3 red, 1 blue and 1 white
(b) 3 red, 2 blue and 1 white
balls if we do not distinguish between balls of the same colour?

Question 4: A box contains 16 balls: 10 white, 4 red and 2 blue balls. We take the balls out of the box one-by-one. In how many different orders can the balls be removed from the box, if we do not distinguish between balls of the same colour?

Question 5: How many different 5-digit numbers can be formed using exactly the digits
(a) 1, 2, 3, 4, 5?
(b) 1, 1, 2, 3, 4?
(c) 1, 1, 2, 2, 2?
(Each digit has to be used exactly as many times as many times it appears in the list.)

Question 6: 15 students are taking part in a running race. How many different outcomes are possible for the first three places, if we assume that there are no equal finishes?

Question 7: In how many different ways can we distribute 6 different books among 10 pupils if everyone can get *at most* one book?

Question 8: How many different 5-digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 6, 7, 8 if
(a) each digit can be used *at most* once?
(b) any digit can be used more than once?

Question 9: How many 6-digit numbers exist in which all digits are different (i.e. which do not contain repeated digits) in the (a) base-10 (b) base-8 (c) base-12 number system?

Question 10: How many different outcomes are possible when
(a) flipping a coin 10 times, (b) rolling a die 10 times,
if the order of the results matters?

Question 11: A multiple choice test consists of 30 questions. For each question 5 possible answers are provided, out of which exactly one answer needs to be selected. In how many different ways can the test be completed?

Question 12: In how many different ways can we distribute 6 identical books among 20 students, if each student can be given *at most* one book?

Question 13: In how many different ways can 4 cards be handed out to a player from a deck of 32 cards? (It does not matter, what order the 4 cards are handed out.)

Question 14: A lottery ticket contains the numbers $1, 2, 3, \ldots, 90$. When filling in the ticket you need to select and mark 5 numbers from among these 90 numbers. (You are trying to guess which 5 numbers will be the winning numbers drawn later.) In how many different ways can you fill in the lottery ticket?

Question 15: Consider the set A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
(a) How many 3-element subsets does A have?
(b) How many 5-element subsets does A have which contains 7?
(c) How many 4-element subsets does A have which contains only odd numbers?
(d) How many subsets does A have in total?

Question 16: We draw 6 cards from a deck of 32 cards (without replacement). How many different outcomes are possible
(a) if the order in which the cards are drawn matters?
(b) if the order in which the cards are drawn does not matter?

Question 17: In how many different ways can we distribute 4 apples among 28 children, if any child can receive more than one apple?

Question 18: In a post office 12 types of cards are sold. In how many different ways can we purchase 5 cards (we assume that the post office has at least 5 copies of each card in stock)?

Question 19: In how many different orders can 4 couples sit along a bench if each person would like to sit next to his/her partner?

Question 20: A company of 8 people would like to sit down at a round table. In how many different orders can they sit around the table, if two particular members of the group: Anna and Ignatious would like to sit next to each other?

Question 21: Given that the number of permutations of $n+2$ (distinct) element equals 20 times the number of permutations of n (distinct) elements, find the value of n.

Further questions

Question 22: Write a program that takes as input a finite set A with $|A| \leq 6$ and prints all permutations without repetition of A.

Question 23: Write a program that takes as input a finite set A with $|A| \leq 6$ and an integer $1 \leq k \leq |A|$, and prints all
(a) k-variations without repetition of A;
(b) k-combinations without repetition of A.

Question 24: Write a program that takes as input a finite set A with $|A| \leq 5$ and an integer $1 \leq k \leq 4$, and prints all k-variations with repetition of A.