Problem set 5:
Complex numbers – algebraic and polar forms of complex numbers

Summary of theory

Algebraic form and basic concepts of complex numbers

complex number, algebraic form: Expressions of the form $a+bi$ where $a, b \in \mathbb{R}$ are called *complex numbers*, where on the set of all complex numbers $\mathbb{C}$ the operations $(+)$ and multiplication $(\cdot)$ are defined as follows: $\forall a+bi, c+di \in \mathbb{C}$:

- $(a+bi)+(c+di) = a+c+(b+d)i$
- $(a+bi)\cdot(c+di) = ac-bd+(ad+bc)i$

The above form $a+bi$ $(a, b \in \mathbb{R})$ is called the *algebraic form* of complex numbers.

real part, imaginary part, conjugate, absolute value: Let $z = a+bi$, where $a, b \in \mathbb{R}$. The *real part of* z is $Re(z) = a$, the *imaginary part of* z is $Im(z) = b$, the *conjugate of* z is $\overline{z} = a-bi$ and the *absolute value of* z is $|z| = \sqrt{a^2+b^2}$.

A method of finding the quotient of two complex numbers in algebraic form: Let $a+bi, c+di \in \mathbb{C}$ be complex numbers, where $c+di \neq 0$. Then the quotient $\frac{a+bi}{c+di}$ can be found in algebraic form "expanding the fraction by the conjugate of the denominator" in the following way:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd+(bc-ad)i}{c^2+d^2} = \underline{\underline{\frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)}{c^2+d^2}i}}$$

Polar form of complex numbers

polar form: The *polar form* of a complex number $z \neq 0$ is $z = r(\cos\varphi + i\cdot\sin\varphi)$, where $r = |z|$.

Note: In the polar form φ is the angle between the position vector of z and the positive real axis in the complex plane. This angle is not unique: $r(\cos\varphi + i\cdot\sin\varphi) = r(\cos(\varphi+2k\pi) + i\cdot\sin(\varphi+2k\pi))$ for every $k \in \mathbb{Z}$.

A method for converting complex numbers from algebraic to polar form: For any $z = a+bi$ $(a, b \in \mathbb{R})$:

$$r = |z| = \sqrt{a^2+b^2} \text{ and } \varphi = \begin{cases} \cos^{-1}\frac{a}{|z|} & \text{if } b \geq 0 \\ -\cos^{-1}\frac{a}{|z|} & \text{if } b < 0 \end{cases}$$

De Moivre's formulas: For any complex numbers $z = |z|(\cos\varphi + i\cdot\sin\varphi)$ and $w = |w|(\cos\psi + i\cdot\sin\psi)$ and any $n \in \mathbb{N}^+$:

1. $zw = |z||w|\left(\cos(\varphi+\psi) + i\cdot\sin(\varphi+\psi)\right)$
2. $\frac{z}{w} = \frac{|z|}{|w|}\left(\cos(\varphi-\psi) + i\cdot\sin(\varphi-\psi)\right)$
3. $z^n = |z|^n\left(\cos(n\varphi) + i\cdot\sin(n\varphi)\right)$

n^{th} roots of a complex number: Let $z \in \mathbb{C}$ and $n \in \mathbb{N}^+$. Then $w \in \mathbb{C}$ is an *n^{th} root of* z if $w^n = z$.

Formula for finding the n^{th} roots of a complex number: Let $z = |z|(\cos\varphi + i \cdot \sin\varphi)$ be arbitrary nonzero complex number and $n \in \mathbb{N}^+$. Then the n^{th} roots of z are:

$$w_k = \sqrt[n]{|z|}\left(\cos\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right) + i \cdot \sin\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right), k = 0, 1, \ldots, n-1.$$

Questions

Algebraic form of complex numbers

Question 1: Calculate the following, giving your answers in algebraic form.

$\sqrt{-16}$ $\qquad$ $\sqrt{-25}$ $\qquad$ $(2i)^2$ $\qquad$ $2i + 5i$ $\qquad$ $\dfrac{4i}{2i}$

Question 2: Let $z \in \mathbb{C}, z = -2 + 7i$. Find the following:

$\operatorname{Re} z$ $\qquad$ $\operatorname{Im} z$ $\qquad$ $-z$ $\qquad$ $\overline{z}$ $\qquad$ $|z|$

Question 3: Calculate the value $\dfrac{4+3i}{(2-i)^2}$ giving your answer in algebraic form.

Question 4: Solve the following equation on the set of complex numbers: $\dfrac{x + i - 3i\overline{x}}{x-4} = i - 1$.

Question 5: Find the complex number(s) $z \in \mathbb{C}$ satisfying the conditions:

$$\left|\frac{z-3}{2-\overline{z}}\right| = 1 \wedge \operatorname{Re}\left(\frac{z}{2+i}\right) = 2$$

Polar form of complex numbers

Question 6: Let $z \in \mathbb{C}, z = 2 + 5i$. Find the absolute value and the argument of z. Represent z on the complex plane (also called Gaussian plane).

Question 7: Write the following complex numbers in polar form:

(a) $1 + i$

(b) $-\sqrt{3} + i$

(c) $\dfrac{9}{2} - \dfrac{9\sqrt{3}}{2}i$

(d) $-\dfrac{\sqrt{14}}{2} - \dfrac{\sqrt{14}}{2}i$

(e) $4i$

(f) i

(g) 10

Question 8: Calculate the following, using the polar form of complex numbers:

(a) $\left(\frac{9}{2}-\frac{9\sqrt{3}}{2}i\right)\left(-\frac{\sqrt{14}}{2}-\frac{\sqrt{14}}{2}i\right)$

(b) $\left(-\frac{3\sqrt{3}}{2}-\frac{3}{2}i\right)\left(\frac{\sqrt{3}}{3}+\frac{1}{3}i\right)$

(c) $\dfrac{-\frac{3\sqrt{3}}{2}-\frac{3}{2}i}{\frac{\sqrt{3}}{3}+\frac{1}{3}i}$

(d) $\left(\frac{5\sqrt{3}}{12}-\frac{5}{12}i\right)^{10}$

(e) $\left(-\frac{\sqrt{10}}{2}-\frac{\sqrt{10}}{2}i\right)^{15}$

(f) $\left(\frac{5}{2}-\frac{5\sqrt{3}}{2}i\right)^{23}$

(g) $(1+i)^8\cdot\left(5\sqrt{3}-5i\right)^3$

(h) $\left(\dfrac{\frac{3}{2}+\frac{3\sqrt{3}}{2}i}{-\frac{5\sqrt{3}}{2}+\frac{5}{2}i}\right)^{12}$

(i) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$

Question 9: Express $z=\dfrac{(1+i)^8}{(1-\sqrt{3}i)^6}$ in algebraic form.

Question 10: Determine the complex roots below:
(a) 2^{nd} roots of -60;
(b) 3^{rd} roots of -60;
(c) 6^{th} roots of $1-\sqrt{3}i$;
(d) 5^{th} roots of $-7\sqrt{3}+7i$;
(e) 8^{th} roots of $-\frac{7}{2}+\frac{7}{2}i$;
(f) 2^{nd} roots of $-6\sqrt{3}+6i$;
(g) 7^{th} roots of $\dfrac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^8}{(1+i)^5}$;

Question 11: Using the polar form of complex numbers, calculate the value of

$$z=\frac{\left(2+2\sqrt{3}i\right)^{10}}{(-1+i)^{83}},$$

giving your answer both in algebraic and in polar forms. Find all complex numbers w such that $w^3=z$, giving your answers in polar form.

Further questions

Question 12: Prove that the addition and multiplication of complex numbers are

(a) commutative and
(b) associative.

Question 13: Prove that the multiplication of complex numbers is distributive over addition, i.e.:

$$\forall v, w, z \in \mathbb{C} : \ v(w+z) = vw + vz \quad (\text{and } (v+w)z = vz + wz)$$

Question 14: Using the definition of multiplication of complex numbers prove that every nonzero complex number z has a multiplicative inverse, i.e. a complex number z^{-1} such that

$$zz^{-1} = z^{-1}z = 1.$$