Problem set 2
Basic concepts and composition of binary relations

Summary of theory

Ordered pair, Cartesian product of sets

The concept of an *ordered pair* (x, y) is defined – using sets – so that $(x, y) = (u, v)$ holds if and only if $x = u$ and $y = v$:

Ordered pair: An *ordered pair* (x, y) is defined as: $(x, y) := \{\{x\}, \{x, y\}\}$. The *first coordinate of* (x, y) is x and the *second coordinate* is y.

Cartesian product: The *Cartesian product* of sets X and Y is $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Basic concepts of binary relations

Binary relation: A subset $R \subseteq X \times Y$ (any subset) of the Cartesian product $X \times Y$ is called a *binary relation from set X to set Y*.

Domain: The *domain* of a binary relation $R \subseteq X \times Y$ is $dmn(R) = \{x \in X \mid \exists y \in Y : (x, y) \in R\}$.

Range: The *range* of a binary relation $R \subseteq X \times Y$ is $rng(R) = \{y \in Y \mid \exists x \in X : (x, y) \in R\}$.

Inverse: The *inverse* of a binary relation R is $R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

Restriction/extension: Let R and S be binary relations. If $R \subseteq S$ then we say that R is a *restriction* of S and S is an *extension* of R.

Restriction to a set: The *restriction of* a relation R *to a set* A is $R|_A = \{(x, y) \mid (x, y) \in R, x \in A\}$.

Image of a set: The *image of* a set A *under a relation* R is $R(A) = \{y \in Y \mid \exists x \in A : (x, y) \in R\}$.

Inverse image: The *inverse image of a set A under a relation R* is $R^{-1}(A) = \{x \in X | \exists y \in A : (x, y) \in R\}$.

Composition of binary relations

Composition of binary relations: The *composition* $R \circ S$ of binary relations R and S is
$R \circ S = \{(x, z) \mid \exists y : \ (x, y) \in S \wedge (y, z) \in R\}$.

Questions

Basic concepts of binary relations

Question 1: Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9\}$. Consider the binary relation

$$\rho = \{(1, 5), (1, 6), (1, 7), (3, 6), (3, 9), (4, 5), (4, 7), (4, 9)\} \subseteq A \times B.$$

(a) Represent ρ on an arrow diagram.
(b) Find the domain and the range of ρ.
(c) Let $S_1 = \{1, 2, 3\}$ and $S_2 = \{4\}$. Determine the restrictions $\rho|_{S_1}$ and $\rho|_{S_2}$ of ρ to sets S_1 and S_2, respectively.
(d) Which of the following relations are extensions of ρ?
$\rho_1 = \{(1, 5), (1, 6), (1, 7), (2, 2), (2, 4), (3, 6), (3, 9), (4, 3), (4, 5), (4, 7), (4, 9)\}$
$\rho_2 = \{(1, 5), (1, 6), (1, 7), (3, 6), (3, 8), (4, 5), (4, 6), (4, 7), (4, 9)\}$
$\rho_3 = A \times B$
$\rho_4 = B \times A$.
(e) Find the inverse ρ^{-1} of ρ, the image $\rho(\{1, 2\})$ and the inverse image $\rho^{-1}(\{5, 6\})$.

Question 2: Define $\rho \subseteq \mathbb{Z} \times \mathbb{Z}$ as $\rho = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 2b\}$. Find $dmn(\rho)$, $rng(\rho)$, ρ^{-1}, the image $\rho(\{3, 4, \dots, 10\})$, the inverse image $\rho^{-1}(\{5, 6, 7, 8\})$ and the restriction $\rho|_{\{1,2,\dots,6\}}$ of ρ to $\{1, 2, \dots, 6\}$.

Question 3: Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y^2 = 2 - x - x^2\}$.
(a) Find the image and the inverse image of set $\{0\}$ under R.
(b) Describe those subsets A of $\mathbb{R}$ for which $R(A)$ contains only one element.
(c) Describe those subsets A of $\mathbb{R}$ for which $R^{-1}(A)$ contains only one element.

Composition of binary relations

Question 4: Let $A = \{1, 2, 3\}, B = \{a, b, c, d, e, f\}$ and $C = \{2, 4, 6, 8\}$. Consider the relations $R = \{(1, a), (1, b), (2, c), (2, f), (3, d), (3, e), (3, f)\} \subseteq A \times B$ and $S = \{(a, 2), (a, 4), (c, 6), (c, 8), (d, 2), (d, 4), (d, 6), (f, 8)\} \subseteq B \times C$. Find the composition $S \circ R$.

Question 5:

(a) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For each of the following relations $S, R \subseteq A \times A$ find $S \circ R$:

(i) $R = \{(1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1)\}$ and $S = \{(1, 6), (2, 3), (2, 4), (3, 1)\}$

(ii) $R = \{(1, 3), (1, 4), (2, 2), (2, 4), (3, 5), (5, 6), (6, 7)\}$ and $S = \{(1, 2), (1, 4), (2, 3), (3, 1), (3, 2), (4, 2), (4, 6), (5, 6), (7, 2)\}$

(iii) $R = \{(2, 2), (2, 4), (3, 1), (3, 4), (4, 4), (5, 3)\}$ and $S = \{(2, 6), (3, 7), (5, 1), (5, 6), (5, 8), (6, 2), (7, 7)\}$

(iv) $R = \{(6, 1), (6, 2), (7, 3), (8, 7)\}$ and $S = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 3), (5, 5), (7, 1), (7, 2)\}$.

(b) Is the composition of relations a commutative operation? (Hint: Determine for example the composition $R \circ S$ in case of (i) in part (a).)

Question 6: For each of the following relations $R, S \subseteq \mathbb{R} \times \mathbb{R}$ find $S \circ R$ and $R \circ S$:

(a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 4x = y^2 + 6\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - 1 = y\}$
(b) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = 2y\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3\}$
(c) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \frac{1}{x} = y^2\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \sqrt{x - 2} = 3y\}$
(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 - 6x + 5 = y\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = y \wedge 2y = x\}$.

Question 7: Let $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = x + 7\}$, $S = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = x - 3\}$ and $T = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x - y \in \{3, 5\}\}$. Find:

$S \circ R \quad R \circ S \quad R^{-1} \circ R \quad R \circ R^{-1} \quad R^3 \quad T \circ R \quad R \circ T \quad T^2 \quad T \circ T^{-1} \quad T^{-1} \circ T$

Question 8: Consider the following relations:
$\rho = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leq 3\}$, $\varphi = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 6x - 1 = 4y + 5\}$,
$\lambda = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 4 \mid 2x + 3y\}$, $\alpha = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 1{,}5x - 1{,}5 \leq y\}$.
Find the compositions:

$\rho \circ \varphi \qquad \varphi \circ \lambda \qquad \varphi^3 \qquad \alpha \circ \rho \qquad \rho \circ \alpha$

Further questions

Question 9: Let $f \subseteq A \times A$ be a binary relation. Prove that $f = f^{-1}$ is true if and only if $f \subseteq f^{-1}$ holds.

Question 10: Let A and B be sets such that $|A| = m$ and $|B| = n$ ($m, n \in \mathbb{N}^+$). How many different binary relations exist from A to B?

Question 11: Write a program that takes as input two arbitrary (finite) binary relations R and S and outputs their compositions $R \circ S$ and $S \circ R$.