Discretization, approximation and optimalization algorithms

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• I. Fekete, L. Lóczi: *Linear multistep methods and global Richardson extrapolation*, Applied Mathematics Letters – **discretization**

• L. Lóczi: *Guaranteed- and high-precision evaluation of the Lambert W function*, Applied Mathematics and Computation – **approximation**

• D. Baráth, L. Hajder, L. Lóczi: *Fast Globally Optimal Surface Normal Estimation from an Affine Correspondence*, submitted – **optimalization**

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Linear multistep methods and global Richardson e

Imre Fekete [a,b,*], Lajos Lóczi [c,d]

(1) Systems of ordinary differential equations with initial values – appearing in modelling

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

Two classes of numerical methods:

• one-step methods, i.e., Runge–Kutta methods (RK)
• linear multistep methods (LMM), i.e., Adams–Bashforth, Adams–Moulton, BDF-methods (for stiff problems)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{j=0}^{k} h\beta_j f_{n+j},$$

(2) Classical Richardson extrapolation (RE): given a sequence depending on a parameter, how can one accelerate its convergence?

Idea: transform the sequence by considering a suitable linear combination of its terms
Two types of RE: global (form the linear combination only at the last step) or local (form the linear combination in every step)

In the literature: RK methods have been combined with RE

Idea: combine LMMs with RE – global version

$$r_n(h) := \frac{2^p}{2^p - 1} \cdot y_{2n}\left(\frac{h}{2}\right) - \frac{1}{2^p - 1} \cdot y_n(h),$$

Here, the sequence y_n is generated by the underlying LMM depending on the discretization step-size h.

Theorem. If the underlying method is of order p, then the extrapolated sequence has order p+1.

Advantage: existing LMM codes can directly be used to implement the LMM+RE method

Theorem. Under some natural assumptions, the region of absolute stability of the LMM+RE method is the same as that of the underlying LMM method.

Table 1
Convergence order and $A(\alpha)$-stability angles for the BDFk-GRE

k	Order	$A(\alpha)$
1	$p = 2$	$90°$,
2	$p = 3$	$90°$,
3	$p = 4$	86.03
4	$p = 5$	73.35
5	$p = 6$	51.83
6	$p = 7$	17.83

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Applied Mathematics and Con

journal homepage: www.elsevier.com/loca

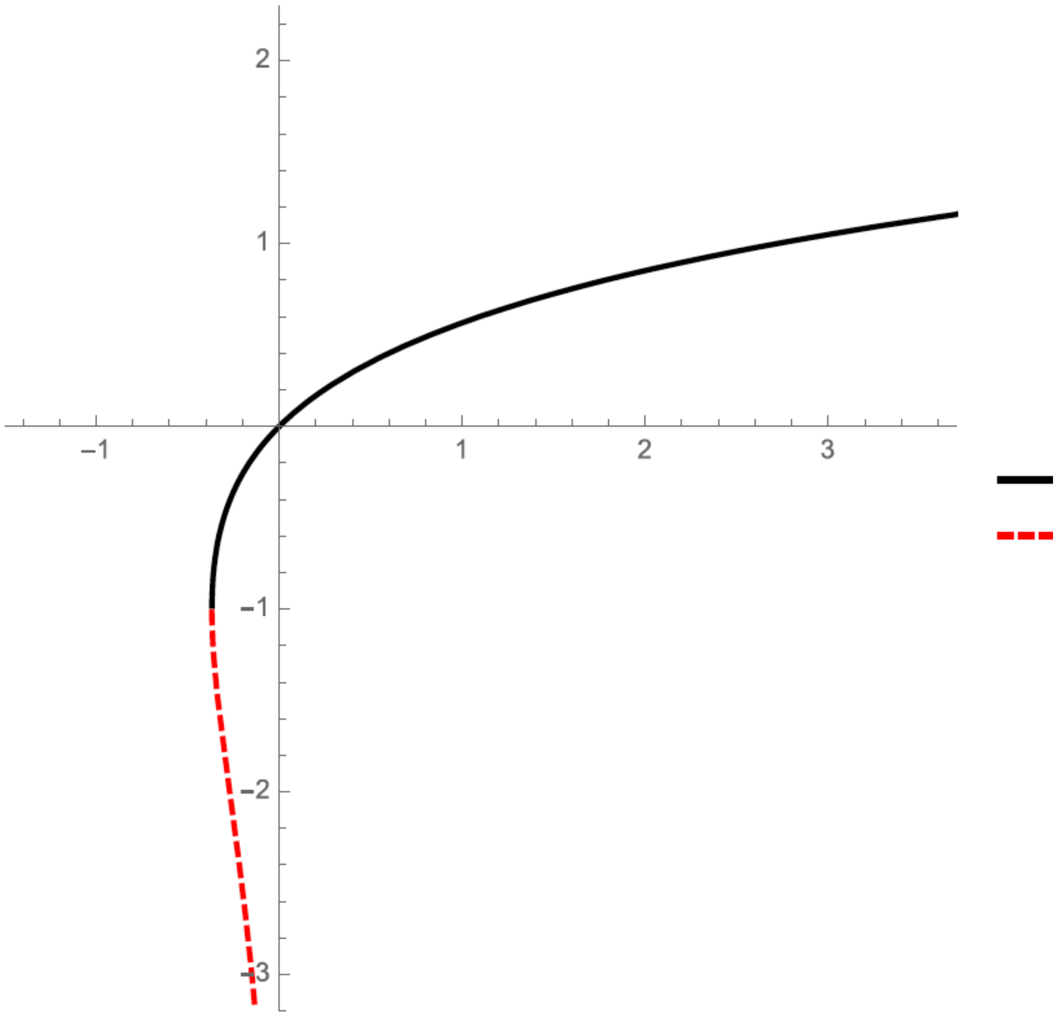
Guaranteed- and high-precision evaluation of th function☆

Lajos Lóczi [a,b]

The Lambert function W satisfies $W(x)\, e^{W(x)} = x$ (for $x > -1/e$) — a generalization of the logarithm function

The solutions to many polynomial-exponential-logarithmic equations can be expressed in terms of the W function

The W function has two real branches: W_0 (continuous curve) and W_{-1} (dashed curve)



The W function gained popularity in the last few decades, and it is implemented in all major symbolic systems (e.g. *Mathematica*, Maple).

Both branches of the W function are now extensively used in science and engineering:

D.A. Barry et al. / Mathematics and Computers in Simulati

Table 1
Applications of the real-valued W-function including the branch used

Problem description	Branch of the W-fun
Water movement in soil	W_{-1} or W_0^- or W_0^+
Enzyme–substrate reactions	W_0^+ or W_0^-
Time of a parachute jump	W_0^+
Iterated exponentiation	$W_0(x)$, $-\exp(-1) \leq$
Jet fuel consumption	W_0^- or W_{-1}
Combustion	W_0^+
Forces in hydrogen ions	W_0^+ or W_0^-
Population growth	W_{-1} and W_0^-
Roots of trinomials	W_0^+
Disease spreading	W_0^-
Recurrences in algorithm analysis	W_0^-
Binary search tree height	W_0^-
Hashing with uniform probing	W_0^+
Hashing methods	W_{-1}
Optimal wire shapes	W_0^-
SU(N) gauge theory	W_0^+
QCD renormalisation	W_0^+ or W_0^- or W_{-1}
Star collapse	W_0^+
Two-body motion	W_0^- and W_{-1}
Structure learning	W_0^+
Reaction–diffusion modelling	W_{-1}
Sample partitioning	W_0^+
Entropy-constrained scalar quantization	W_0^-
Redox barrier design	W_0^-
Photochemical bleaching	W_0^+
Thin film life time	W_{-1}
Testing Legendre transform algorithm	W_0^+
Exponential function approximation	W_{-1} and W_0^-
Herbivore–plant coexistence	W_0^+
Photorefractive two-wave mixing	W_0^+

The W function is not an elementary function, natural question: how to approximate it with elementary functions?
There are several known formulae, including

- Taylor expansions, e.g., about the origin

$$\sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k = x - x^2 + \frac{3x^3}{2} - \frac{8x^4}{3} +$$

- Puiseux expansions, e.g., about the branch point $x =$

- asymptotic expansions about $+\infty$, such as

$$\ln(x) - \ln(\ln(x)) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{k,m} \frac{(\ln}{(l}$$

where the coefficients $c_{k,m}$ are defined in terms of th

recursive approximations

- the recursion

$$\lambda_{n+1}(x) := \ln(x) - \ln(\lambda_n(x$$

- the Newton-type iteration

$$\nu_{n+1}(x) := \nu_n(x) - \frac{\nu_n(x) - xe}{1 + \nu_n(}$$

- the iteration

$$\beta_{n+1}(x) := \frac{\beta_n(x)}{1 + \beta_n(x)} \left(1 + \ln \left(\frac{}{\beta}\right.\right.$$

- the Halley-type iteration

$$h_{n+1}(x) := h_n(x) - \frac{h_n(x) e^{h_n(x}}{e^{h_n(x)} (h_n(x) + 1) - \frac{(h_n}{}}$$

- the Fritsch–Shafer–Crowley (FSC) scheme;

analytic bounds on different intervals

- the bounds

$$\ln(x) - \ln(\ln(x)) + \frac{\ln(\ln(x))}{2\ln(x)} < W_0(x) < \ln(x) - \mathrm{l}$$

valid for $x \in (e, +\infty)$;

Error estimates for the remainder terms in the series expansions?

For the recursive approximations:
What starting value should one pick?
Is the recursion well-defined then?
Will it converge for a particular value of x?
If yes, what is the error committed when n recursive steps are performed?
How many steps to take to approximate $W(x)$ to a given precision?
How to tackle the difficulties when x is close to the branch point at $-1/e$, to the singularity near $x < 0$, or when $x > 0$ is very large?

In our work, we analyzed the following recursion proposed by R. Iacono and J. P. Boyd:

$$\beta_{n+1}(x) := \frac{\beta_n(x)}{1+\beta_n(x)}\left(1 + \ln\left(\frac{x}{\beta_n(x)}\right)\right)$$

• We proposed **simple and suitable starting values** (consisting of the basic operations, logarithms, or square roots) that guarantee monotone convergence on the full domain of definition of both real branches.

• The quadratic rate of convergence of the above recursion is proved via **explicit** and **uniform** error estimates.

• From these estimates, the maximum number of iteration steps needed to achieve a desired precision can easily be determined in advance.

WIKIPEDIA
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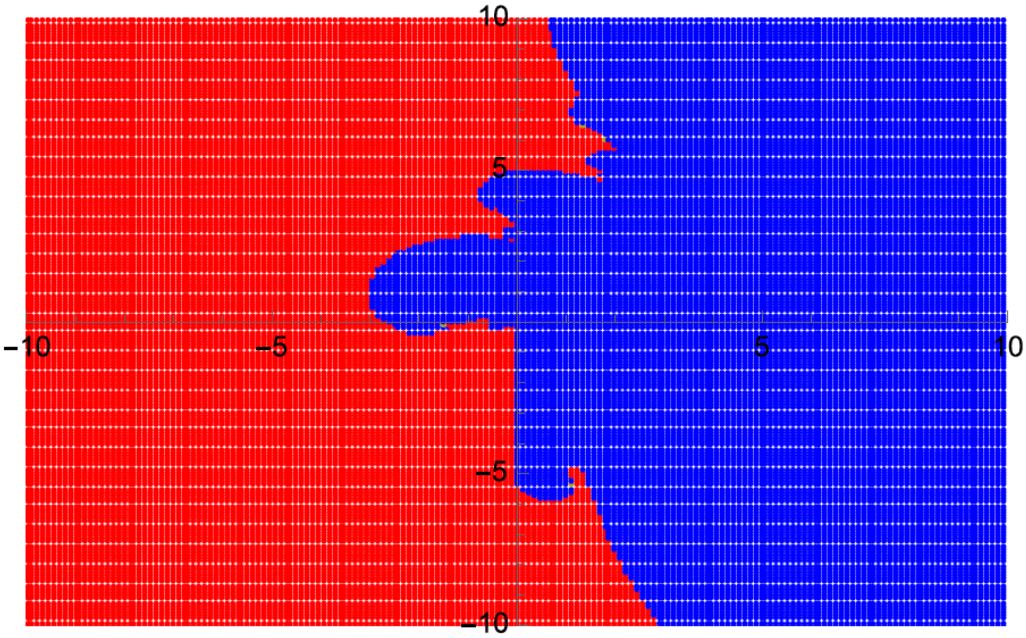
Article Talk

Lambert *W* function

From Wikipedia, the free encyclopedia

• Open questions: the W function has also ∞ many complex branches. Can one develop suitable recursions to approximate these + guaranteed error bounds?

• Main difficulty: what starting value to choose?



• Possible topic for a PhD dissertation

Fast Globally Optimal Surface Normal Estimatio
Correspondence

Anonymous CVPR submission

Paper ID 9196

Abstract

We introduce a new, globally optimal solver for estimating the surface normal from a single affine correspondence in two calibrated views. In contrast to previous solutions to this problem, the proposed algorithm contains an explicit formula which has been obtained by solving a linear equation, thus, it is significantly faster than its similarly accurate alternatives. We show on 15k image pairs from standard benchmarks that the proposed approach leads to exactly the same results as other optimal algorithms while being, on average, five times faster than the fastest alternative. Besides its theoretical value, we demonstrate that such an approach has clear benefits, e.g., in image-based visual localization, due to not requiring a dense point cloud to recover the surface normal. We show on the Cambridge Landmarks dataset that leveraging the normals estimated by the proposed algorithm further improves the localization accuracy. The code will be made publicly available.

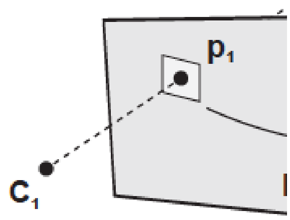


Figure 1. **Geometric co**
with normal $\mathbf{n} \in \mathbb{E}^3$ pr
Π^2 as $\mathbf{p}_2 \in \mathbb{E}^2$. The loc
related by the local affine

terparts [63]. Moreov
the numerical refinem
point cloud [26]. In thi
lem, *i.e.*, estimating th

$$\mathbf{n}^* = \arg\min_{\mathbf{n}} f(\mathbf{n}),$$

with

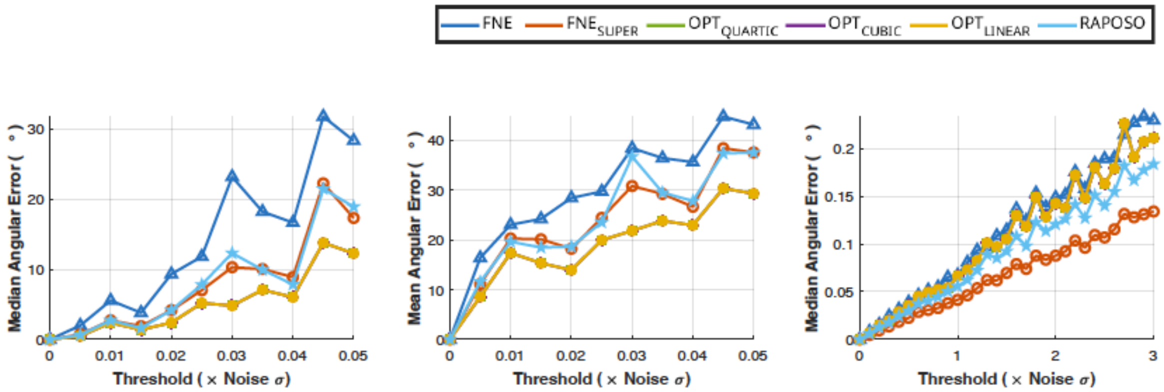
$$f(\mathbf{n}) := \sum_{i=1}^{2} \sum_{j=1}^{2} \left(\frac{\mathbf{n}^\top \mathbf{w}_{ij}}{\mathbf{n}^\top \mathbf{w}_c} - a_{ij} \right)^2 .$$

Solutions:

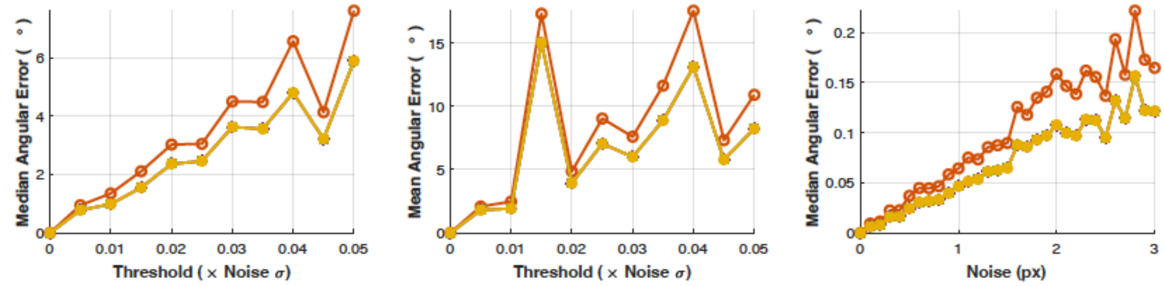
• 2015: a globally optimal estimator in the least squares sense as the solution of a quartic polinomial
• 2020: the optimal solution can also be found as a real root of a cubic polynomial

• In our work: we propose a new **linear** and **globally optimal** solution to surface normal estimation from a single affine correspondence. The derived symbolic formula contains 19 parameters, and consists of **only the four basic arithmetic operations**. Therefore, one can completely avoid root extractions, solving any higher degree polynomial equations, or employing any numerical or iterative methods. This new algorithm is **five times faster** than the above cubic solver.

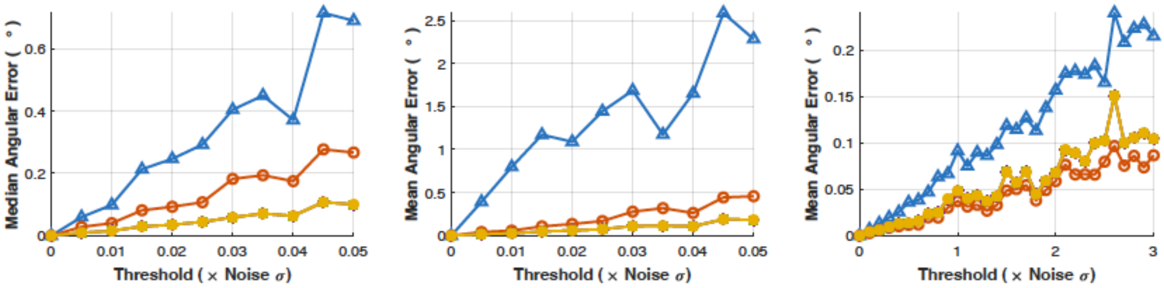
Extensive tests and comparisons:



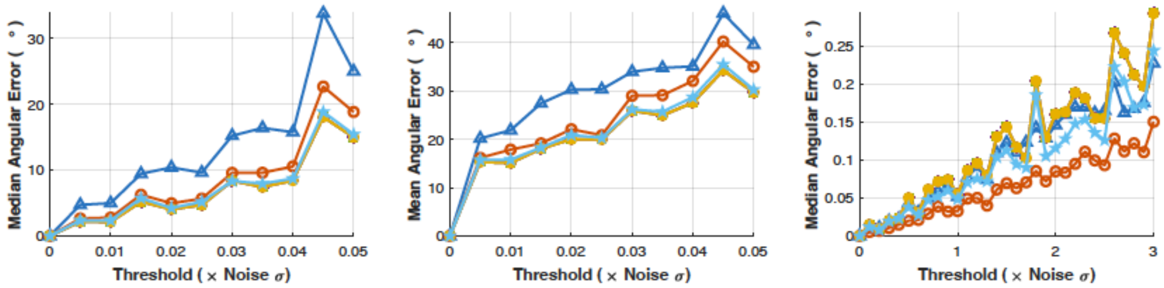
(a) General Motion.



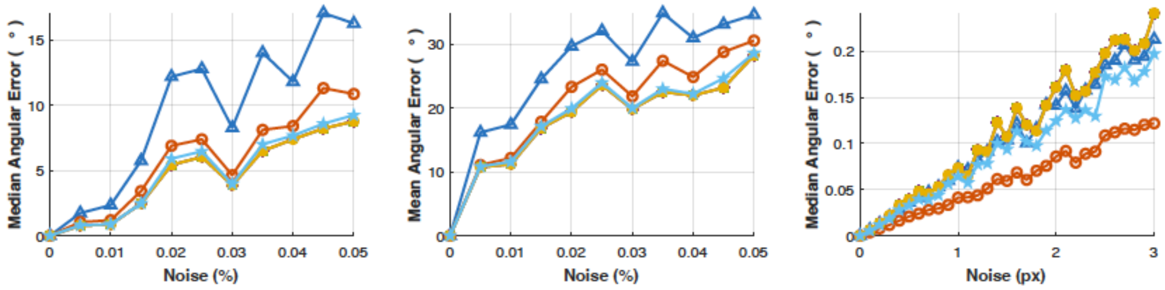
(b) Standard Stereo.



(c) Planar Motion.



(d) Moto Motion.



(e) Drone Motion.

The actual formulae:

$$
\begin{aligned}
\mathrm{numer}_1 := {} & a_{11} w_{11,x} w_{c,y}^2 n_y^2 + a_{12} w_{12,x} w_{c,y}^2 n_y^2 + a_{21} w_{21,x} w_{c,y}^2 n_y^2 + a_{22} w_{22,x} \\
& w_{12,y}^2 w_{c,x} n_y^2 + w_{21,y}^2 w_{c,x} n_y^2 + w_{22,y}^2 w_{c,x} n_y^2 - w_{11,x} w_{11,y} w_{c,y} n_y^2 - w_{12,x} \\
& w_{22,x} w_{22,y} w_{c,y} n_y^2 - a_{11} w_{11,y} w_{c,x} w_{c,y} n_y^2 - a_{12} w_{12,y} w_{c,x} w_{c,y} n_y^2 - a_{21} w_{21} \\
& 2 w_{11,y} w_{11,z} w_{c,x} n_y + 2 w_{12,y} w_{12,z} w_{c,x} n_y + 2 w_{21,y} w_{21,z} w_{c,x} n_y + 2 w_{22,y} \\
& w_{12,x} w_{12,z} w_{c,y} n_y - w_{21,x} w_{21,z} w_{c,y} n_y - w_{22,x} w_{22,z} w_{c,y} n_y - a_{11} w_{11,z} \\
& a_{21} w_{21,z} w_{c,x} w_{c,y} n_y - a_{22} w_{22,z} w_{c,x} w_{c,y} n_y - w_{11,x} w_{11,y} w_{c,z} n_y - w_{12,x} \\
& w_{22,x} w_{22,y} w_{c,z} n_y - a_{11} w_{11,y} w_{c,x} w_{c,z} n_y - a_{12} w_{12,y} w_{c,x} w_{c,z} n_y - a_{21} w_{21} \\
& 2 a_{11} w_{11,x} w_{c,y} w_{c,z} n_y + 2 a_{12} w_{12,x} w_{c,y} w_{c,z} n_y + 2 a_{21} w_{21,x} w_{c,y} w_{c,z} n_y + 2 \\
& a_{12} w_{12,x} w_{c,z}^2 + a_{21} w_{21,x} w_{c,z}^2 + a_{22} w_{22,x} w_{c,z}^2 + w_{11,z}^2 w_{c,x} + w_{12,z}^2 w_{c,x} + w_{2} \\
& w_{12,x} w_{12,z} w_{c,z} - w_{21,x} w_{21,z} w_{c,z} - w_{22,x} w_{22,z} w_{c,z} - a_{11} w_{11,z} w_{c,x} w_{c,z} - a
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{denom}_1 := {} & -a_{11} n_y w_{11,x} w_{c,x} w_{c,y} + a_{11} n_y w_{11,y} w_{c,x}^2 + a_{12} n_y w_{12,y} w_{c,x}^2 + a_{21} n \\
& a_{22} n_y w_{22,y} w_{c,x}^2 - a_{12} n_y w_{12,x} w_{c,x} w_{c,y} - a_{21} n_y w_{21,x} w_{c,x} w_{c,y} - a_{22} n_y w_{22} \\
& a_{11} w_{11,z} w_{c,x}^2 + a_{12} w_{12,z} w_{c,x}^2 + a_{21} w_{21,z} w_{c,x}^2 + a_{22} w_{22,z} w_{c,x}^2 - a_{12} w_{12,x} \\
& a_{22} w_{22,x} w_{c,x} w_{c,z} + n_y w_{11,x}^2 w_{c,y} - n_y w_{11,x} w_{11,y} w_{c,x} - n_y w_{12,x} w_{12} \\
& n_y w_{22,x} w_{22,y} w_{c,x} + n_y w_{12,x}^2 w_{c,y} + n_y w_{21,x}^2 w_{c,y} + n_y w_{22,x}^2 w_{c,y} + w_{1} \\
& w_{12,x} w_{12,z} w_{c,x} - w_{21,x} w_{21,z} w_{c,x} - w_{22,x} w_{22,z} w_{c,x} -
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{numer}_2 := {} & ((a_{12} w_{12,y} + a_{21} w_{21,y} + a_{22} w_{22,y}) w_{c,z}^2 - (w_{12,y} w_{12,z} + a_{12} w_{c,y} w_{1} \\
& w_{22,y} w_{22,z} + a_{21} w_{21,z} w_{c,y} + a_{22} w_{22,z} w_{c,y}) w_{c,z} + (w_{12,z}^2 + w_{21,z}^2 + w_{22,z}^2) \\
& a_{11} w_{12,y} w_{12,x} + a_{21} w_{11,y} w_{21,x} + a_{11} w_{21,x} w_{21,y} + a_{22} w_{11,y} w_{22,x} \\
& w_{11,y} (w_{12,x} w_{12,z} + a_{12} w_{c,x} w_{12,z} + w_{21,x} w_{21,z} + w_{22,x} w_{22,z} + a_{21} w_{21} \\
& a_{11} (w_{12,y} w_{12,z} w_{c,x} + w_{21,y} w_{21,z} w_{c,x} + w_{22,y} w_{22,z} w_{c,x} + w_{12,x} w_{12} \\
& w_{22,x} w_{22,z} w_{c,y}) w_{c,z} - (w_{12,z}^2 + w_{21,z}^2 + w_{22,z}^2) w_{c,x} (w_{11,y} + a_{11} w_{c,y})) \\
& a_{11} w_{11,y} w_{21,z}^2 w_{c,x}^2 + a_{12} w_{12,y} w_{21,z}^2 w_{c,x}^2 + a_{11} w_{11,y} w_{22,z}^2 w_{c,x}^2 + a_{12} w_{12,y}
\end{aligned}
$$

$$a_{21} w_{12,z}^2 w_{21,y} w_{c,x}^2 - a_{21} w_{12,y} w_{12,z} w_{21,z} w_{c,x}^2 - a_{12} w_{12,z} w_{21,y} w_{21,z} u$$

$$a_{22} w_{21,z}^2 w_{22,y} w_{c,x}^2 - a_{22} w_{12,y} w_{12,z} w_{22,z} w_{c,x}^2 - a_{22} w_{21,y} w_{21,z} w_{22,z} w_{c,x}^2$$

$$a_{21} w_{21,z} w_{22,y} w_{22,z} w_{c,x}^2 + (a_{21} w_{21,y} w_{12,x}^2 - a_{21} w_{12,y} w_{21,x} w_{12,x} - a_{12} w_{21,x}$$

$$a_{12} w_{12,y} w_{21,x}^2 + a_{12} w_{12,y} w_{22,x}^2 + a_{21} w_{21,y} w_{22,x}^2 - a_{22} w_{21,x} w_{21,y} w_{22,x} + a$$

$$a_{22}(w_{12,x}^2 + w_{21,x}^2) w_{22,y} - (a_{12} w_{12,x} + a_{21} w_{21,x}) w_{22,x} w_{22,y}) w_{c,z}^2$$

$$w_{12,x} w_{12,y} w_{22,z}^2 w_{c,x} - w_{21,x} w_{21,y} w_{22,z}^2 w_{c,x} - w_{12,z}^2 w_{21,x} w_{21,y} w_{c,x} +$$

$$w_{12,x} w_{12,z} w_{21,y} w_{21,z} w_{c,x} - w_{12,z}^2 w_{22,x} w_{22,y} w_{c,x} - w_{21,z}^2 w_{22,x} w_{22,y} w_{c,x}$$

$$w_{21,y} w_{21,z} w_{22,x} w_{22,z} w_{c,x} + w_{12,x} w_{12,z} w_{22,y} w_{22,z} w_{c,x} + w_{21,x} w_{21,z} w_{22,}$$

$$w_{12,x}^2 w_{21,z}^2 w_{c,y} + w_{12,z}^2 w_{22,x}^2 w_{c,y} + w_{21,z}^2 w_{22,x}^2 w_{c,y} + w_{12,x}^2 w_{22,z}^2$$

$$2 w_{12,x} w_{12,z} w_{21,x} w_{21,z} w_{c,y} - 2 w_{12,x} w_{12,z} w_{22,x} w_{22,z} w_{c,y} - 2 w_{21,}$$

$$a_{12} w_{12,x} w_{21,z}^2 w_{c,x} w_{c,y} - a_{12} w_{12,x} w_{22,z}^2 w_{c,x} w_{c,y} - a_{21} w_{21,x} w_{22,z}^2 w_{c,x} w$$

$$a_{21} w_{12,x} w_{12,z} w_{21,z} w_{c,x} w_{c,y} + a_{12} w_{12,z} w_{21,x} w_{21,z} w_{c,x} w_{c,y} - a_2$$

$$a_{22} w_{21,z}^2 w_{22,x} w_{c,x} w_{c,y} + a_{22} w_{12,x} w_{12,z} w_{22,z} w_{c,x} w_{c,y} + a_{22} w_{21,}$$

$$a_{12} w_{12,z} w_{22,x} w_{22,z} w_{c,x} w_{c,y} + a_{21} w_{21,z} w_{22,x} w_{22,z} w_{c,x} w_{c,y} + w_{11,z}^2((a_{12} w$$

$$-(w_{12,x} w_{12,y} + w_{21,x} w_{21,y} + w_{22,x} w_{22,y}) w_{c,x} + (w_{12,x}^2 + w_{21,x}^2 + w_2^2$$

$$a_{22} w_{22,x}) w_{c,x}) w_{c,y}) + (-((w_{21,y} w_{21,z} + a_{21} w_{c,y} w_{21,z} + w_{22,y} w_{22,}$$

$$(a_{12}(w_{21,y} w_{21,z} w_{c,x} + w_{22,y} w_{22,z} w_{c,x} + w_{21,x} w_{21,z} w_{c,y} + w_{22,x} w_{22,z} w_{c,y})$$

$$a_{22} w_{22,y}) w_{c,x} + w_{21,x}(w_{21,y} + a_{21} w_{c,y}) + w_{22,x}(w_{22,y} + a_{22} w_{c,y})))$$

$$w_{21,x} w_{21,z} w_{22,x} w_{22,y} + w_{21,x} w_{21,y} w_{22,x} w_{22,z} - w_{21,x}^2 w_{22,y} w_{22,z} +$$

$$2 a_{11} w_{11,y} w_{21,x} w_{21,z} w_{c,x} + a_{22} w_{21,y} w_{21,z} w_{22,x} w_{c,x} - 2 a_{22} w_{21,x} w_{21,z} w_{22,y}$$

$$a_{21} w_{21,z} w_{22,x} w_{22,y} w_{c,x} + a_{22} w_{21,x} w_{21,y} w_{22,z} w_{c,x} - 2 a_{11} w_{11,y} w_{22,x} w_{22,z} u$$

$$a_{21} w_{21,x} w_{22,y} w_{22,z} w_{c,x} + w_{12,y}(-w_{12,z}(w_{21,x}^2 + w_{22,x}^2) + w_{12,z}(a_2$$

$$2 a_{12}(w_{21,x} w_{21,z} + w_{22,x} w_{22,z}) w_{c,x} + w_{12,x}(w_{21,x} w_{21,z} + a_{21} w_{c,x} w_{21,z} +$$

$$(a_{12} w_{12,z}(w_{21,x}^2 + w_{22,x}^2) + (a_{22} w_{21,x} - a_{21} w_{22,x})(w_{21,x} w_{22,z} -$$

$$w_{11,z}(w_{11,y} w_{c,x}(w_{12,x} w_{12,z} + w_{21,x} w_{21,z} + w_{22,x} w_{22,z} - (a_{12} w_{12,z} +$$

$$a_{11} w_{c,x}((w_{12,x} w_{12,z} + w_{21,x} w_{21,z} + w_{22,x} w_{22,z}) w_{c,y} - (w_{12,y} w_{12,z} + w$$

$$w_{11,y}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12} w_{12,x} + a_{21} w_{21,x} + a_{22} w_{22,x}) w_{c,}$$

$$w_{21,x} w_{21,y} + w_{22,x} w_{22,y}) w_{c,x} - (w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2) w_{c,y}) w_{c,z}$$

$$w_{21,y} w_{21,z} w_{c,x} + w_{22,y} w_{22,z} w_{c,x} + a_{12} w_{12,z} w_{c,y} w_{c,x} + a_{21} w_{21,z} w_{c,y}$$

$$2 w_{12,x} w_{12,z} w_{c,y} - 2 w_{21,x} w_{21,z} w_{c,y} - 2 w_{22,x} w_{22,z} w_{c,y} + (w_{21,x} w_{21,y} + w_{22}$$

$$a_{22} w_{22,y}) w_{c,x} + (a_{21} w_{21,x} + a_{22} w_{22,x}) u$$

$$
\begin{aligned}
\text{denom}_2 := {} & ((a_{12}w_{12,z} + a_{21}w_{21,z} + a_{22}w_{22,z})w_{c,y}^2 - (w_{12,y}w_{12,z} + w_{21,y}w_{21,} \\
& (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2 - (a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,y})w_{c,z}) \\
& w_{21,y}w_{21,z} + w_{22,y}w_{22,z})w_{c,x} - (w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,} \\
& a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2) + (w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} + w_{21,x}w_{21,y} \\
& a_{22}w_{22,y}w_{c,x})w_{c,y} - (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x}) + a_{11}((w_{12,x}w_{12,y} + \\
& (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x})w_{c,z})w_{11,x} + a_{11}w_{11,z}w_{12,y}^2w_{c,x}^2 + a_{11}w_{11,z} \\
& a_{11}w_{11,z}w_{22,y}^2w_{c,x}^2 + a_{12}w_{12,z}w_{22,y}^2w_{c,x}^2 + a_{21}w_{21,z}w_{22,y}^2w_{c,x}^2 - \\
& a_{21}w_{12,y}^2w_{21,z}w_{c,x}^2 - a_{12}w_{12,y}w_{21,y}w_{21,z}w_{c,x}^2 - a_{22}w_{12,y}w_{12,z}w_{22,y}w_{c}^2 \\
& a_{22}w_{12,y}^2w_{22,z}w_{c,x}^2 + a_{22}w_{21,y}^2w_{22,z}w_{c,x}^2 - a_{12}w_{12,y}w_{22,y}w_{22,z}w_{c,x}^2 \\
& a_{11}w_{11,z}w_{12,x}^2w_{c,y}^2 + a_{11}w_{11,z}w_{21,x}^2w_{c,y}^2 + a_{12}w_{12,z}w_{21,x}^2w_{c,y}^2 + a_{11}w_{11,z} \\
& a_{21}w_{21,z}w_{22,x}^2w_{c,y}^2 - a_{21}w_{12,x}w_{12,z}w_{21,x}w_{c,y}^2 + a_{21}w_{12,x}^2w_{21,z}w_{c,y}^2 \\
& a_{22}w_{12,x}w_{12,z}w_{22,x}w_{c,y}^2 - a_{22}w_{21,x}w_{21,z}w_{22,x}w_{c,y}^2 + a_{22}w_{12,x}^2w_{22,z} \\
& a_{12}w_{12,x}w_{22,x}w_{22,z}w_{c,y}^2 - a_{21}w_{21,x}w_{22,x}w_{22,z}w_{c,y}^2 - w_{12,x}w_{12,z}w_{21,y}^2 \\
& w_{21,x}w_{21,z}w_{22,y}^2w_{c,x} + w_{12,y}w_{12,z}w_{21,x}w_{21,y}w_{c,x} - w_{12,y}^2w_{21,x}w_{21,z}w_{c,} \\
& w_{12,y}w_{12,z}w_{22,x}w_{22,y}w_{c,x} + w_{21,y}w_{21,z}w_{22,x}w_{22,y}w_{c,x} - w_{12,y}^2w_{22,x}w_{2} \\
& w_{12,x}w_{12,y}w_{22,y}w_{22,z}w_{c,x} + w_{21,x}w_{21,y}w_{22,y}w_{22,z}w_{c,x} - w_{12,y}w_{12,z}w_{21}^2 \\
& w_{21,y}w_{21,z}w_{22,x}^2w_{c,y} + w_{12,x}w_{12,z}w_{21,x}w_{21,y}w_{c,y} + w_{12,x}w_{12,y}w_{21,x}w_{2} \\
& w_{12,x}w_{12,z}w_{22,x}w_{22,y}w_{c,y} + w_{21,x}w_{21,z}w_{22,x}w_{22,y}w_{c,y} + w_{12,} \\
& w_{21,x}w_{21,y}w_{22,x}w_{22,z}w_{c,y} - w_{12,x}^2w_{22,y}w_{22,z}w_{c,y} - w_{21,x}^2w_{22,y}w_{22,z}w_{c,y} \\
& a_{21}w_{12,y}w_{12,z}w_{21,x}w_{c,x}w_{c,y} + a_{21}w_{12,x}w_{12,z}w_{21,y}w_{c,x}w_{c,y} - 2a_{11} \\
& 2a_{12}w_{12,z}w_{21,x}w_{21,y}w_{c,x}w_{c,y} - 2a_{21}w_{12,x}w_{12,y}w_{21,z}w_{c,x}w_{c,y} + a_{1} \\
& a_{12}w_{12,x}w_{21,y}w_{21,z}w_{c,x}w_{c,y} + a_{22}w_{12,y}w_{12,z}w_{22,x}w_{c,x}w_{c,y} + a_{22} \\
& a_{22}w_{12,x}w_{12,z}w_{22,y}w_{c,x}w_{c,y} + a_{22}w_{21,x}w_{21,z}w_{22,y}w_{c,x}w_{c,y} - 2a_{11} \\
& 2a_{12}w_{12,z}w_{22,x}w_{22,y}w_{c,x}w_{c,y} - 2a_{21}w_{21,z}w_{22,x}w_{22,y}w_{c,x}w_{c,y} - 2a_{2} \\
& 2a_{22}w_{21,x}w_{21,y}w_{22,z}w_{c,x}w_{c,y} + a_{12}w_{12,y}w_{22,x}w_{22,z}w_{c,x}w_{c,y} + a_{21} \\
& a_{12}w_{12,x}w_{22,y}w_{22,z}w_{c,x}w_{c,y} + a_{21}w_{21,x}w_{22,y}w_{22,z}w_{c,x}w_{c,y} + ((w \\
& a_{22}w_{22,y})w_{c,y})w_{12,x}^2 + a_{12}((w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,y} - (w_{2}^2 \\
& w_{12,y}^2(w_{21,x}^2 + w_{22,x}^2 - (a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x}) + (w_{21,y}w_{22,x} - \\
& a_{21}w_{c,y}w_{22,x} - w_{21,x}w_{22,y} - a_{22}w_{21,y}w_{c,x} + a_{21}w_{22,y}w_{c,x} + a_{22}w_{21,x} \\
& w_{22,x}w_{22,y})w_{c,x} - a_{12}(w_{21,x}^2 + w_{22,x}^2)w_{c,y} + w_{12,x}(-2w_{21,x}w_{21,y} + \\
& a_{22}w_{22,y}w_{c,x} + a_{21}w_{21,x}w_{c,y} + a_{22}w_{22,x}w_{c,y})))w_{c,z} + w_{11,y}^2((a_{12}w_{12,z} \\
& (w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,x} + (w_{12,x}^2 + w_{21,x}^2 + w_{2}^2
\end{aligned}
$$

$$a_{22}w_{22,x})w_{c,x})w_{c,z}) + w_{11,y}(w_{11,z}w_{c,x}(w_{12,x}w_{12,y} + w_{21,x}w_{21,y} +$$
$$a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x}) - w_{11,z}(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 - (a_{12}w_{12,x} +$$
$$a_{11}w_{c,x}((w_{12,x}w_{12,z} + w_{21,x}w_{21,z} + w_{22,x}w_{22,z})w_{c,y} - (w_{12,y}w_{12,z} +$$
$$a_{11}((w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} - (w_{12,x}^2 + w_2^2$$
$$w_{11,x}(w_{12,y}w_{12,z}w_{c,x} + w_{21,y}w_{21,z}w_{c,x} + w_{22,y}w_{22,z}w_{c,x} - 2a_{12}w_{12,z}$$
$$2a_{22}w_{22,z}w_{c,y}w_{c,x} + a_{12}w_{12,y}w_{c,z}w_{c,x} + a_{21}w_{21,y}w_{c,z}w_{c,x} + a_{22}w_{22,}$$
$$w_{21,x}w_{21,z}w_{c,y} + w_{22,x}w_{22,z}w_{c,y} - 2w_{12,x}w_{12,y}w_{c,z} - 2w_{21,x}w_{21,}$$
$$a_{12}w_{12,x}w_{c,y}w_{c,z} + a_2$$

$$\text{numer}_3 := n_y((a_{11}w_{11,x} + a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2 -$$
$$w_{c,y}((a_{11}w_{11,y} + a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})w_{c,x} + w_{11,x}w_{11,y} + w_{1}$$
$$w_{22,x}w_{22,y}) + (w_{11,y}^2$$

$$\text{denom}_3 := (a_{11}w_{11,y} + a_{12}w_{12,y} + a_{21}w_{21,y} + a_{22}w_{22,y})\, w_{c,x}^2 +$$
$$w_{c,y}\,(-\,(a_{11}w_{11,x} + a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})\, w_{c,x} + w_{11,x}^2 + w_{12}^2$$
$$w_{c,x}\,(w_{11,x}w_{11,y} + w_{12,x}w_1$$

$$\text{numer}_4 :=$$
$$(w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2 + (a_{12}^2 + a_{21}^2 + a_{22}^2)w_{c,y}^2 - 2(a_{12}w_{12,y} + a_{21}w_{21,y} +$$
$$2a_{11}(-(a_{12}w_{12,x} + a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,y}^2 + (w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} -$$
$$+ a_{21}w_{21,y}w_{c,x} + a_{22}w_{22,y}w_{c,x})w_{c,y} - (w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2)w_{c,x})w_{11,x} +$$
$$w_{12,y}^2w_{22,x}^2 + w_{21,y}^2w_{22,x}^2 + w_{12,x}^2w_{22,y}^2 + w_{21,x}^2w_{22,y}^2 + a_{11}^2w_{12,y}^2w_{c,x}^2 + a_{21}^2w_1^2$$
$$a_{11}^2w_{21,y}^2w_{c,x}^2 + a_{12}^2w_{21,y}^2w_{c,x}^2 + a_{22}^2w_{21,y}^2w_{c,x}^2 + a_{11}^2w_{22,y}^2w_{c,x}^2 + a_{12}^2w_{22,y}^2$$
$$2a_{12}a_{21}w_{12,y}w_{21,y}w_{c,x}^2 - 2a_{12}a_{22}w_{12,y}w_{22,y}w_{c,x}^2 - 2a_{21}a_{22}w_{21,y}w_{22,y}w_{c,x}^2 + (($$
$$a_{12}^2w_{21,x}^2 + a_{12}^2w_{22,x}^2 + a_{22}^2(w_{12,x}^2 + w_{21,x}^2) - 2a_{12}a_{22}w_{12,x}w_{22,x} - 2a_{21}w_{21,x}$$
$$a_{21}^2(w_{12,x}^2 + w_{22,x}^2))w_{c,y}^2 - 2w_{12,x}w_{12,y}w_{21,x}w_{21,y} - 2w_{12,x}w_{12,y}w_{22,x}w_{22,y} -$$
$$2a_{12}w_{12,x}w_{21,y}^2w_{c,x} - 2a_{12}w_{12,x}w_{22,y}^2w_{c,x} - 2a_{21}w_{21,x}w_{22,y}^2w_{c,x} - 2a$$
$$2a_{21}w_{12,x}w_{12,y}w_{21,y}w_{c,x} + 2a_{12}w_{12,y}w_{21,x}w_{21,y}w_{c,x} - 2a_{22}w_{12,y}^2w_{22,x}w_{c,x}$$
$$2a_{22}w_{12,x}w_{12,y}w_{22,y}w_{c,x} + 2a_{22}w_{21,x}w_{21,y}w_{22,y}w_{c,x} + 2a_{12}w_{12,y}$$

$$2a_{21}w_{21,y}w_{22,x}w_{22,y}w_{c,x} + w_{11,y}^2(w_{12,x}^2 + w_{21,x}^2 + w_{22,x}^2 + (a_{12}^2 + a_{21}^2 + a$$
$$a_{21}w_{21,x} + a_{22}w_{22,x})w_{c,x}) - 2(w_{22,x}w_{22,y}w_{c,x}a_{11}^2 - a_{22}(w_{12,x}w_{12,y} +$$
$$a_{22}(w_{12,x}^2 + w_{21,x}^2)w_{22,y} + (a_{11}^2 + a_{22}^2)(w_{12,x}w_{12,y} + w_{21,x}w_{21,y})w_{c,x}$$
$$w_{22,x}w_{22,y})w_{c,x} + a_{12}^2(w_{21,x}w_{21,y} + w_{22,x}w_{22,y})w_{c,x} + a_{21}(w_{21,y}w_{12,x}^2$$
$$w_{21,y}w_{22,x}^2 - w_{21,x}w_{22,x}w_{22,y} - (a_{12}w_{12,y}w_{21,x} + a_{22}w_{22,y}w_{21,x} + a_{12}w_{12,x}w_2$$
$$a_{12}w_{12,y}(w_{21,x}^2 + w_{22,x}(w_{22,x} - a_{22}w_{c,x})) - a_{12}w_{12,x}(w_{21,x}w_{21,y} + w_{22,y}($$
$$2w_{11,y}((w_{11,x} - a_{11}w_{c,x})(-w_{12,x}w_{12,y} + a_{12}w_{c,x}w_{12,y} - w_{21,x}w_{21,y} - w_{22,}$$
$$a_{22}w_{22,y}w_{c,x}) + (-w_{11,x}w_{c,x}a_{12}^2 + w_{12,x}(w_{11,x} + a_{11}w_{c,x})a_{12} - a_{11}(w_{12,x}^2 + w_2$$
$$a_{22}w_{22,x})w_{c,x} + w_{11,x}(a_{21}w_{21,x} + a_{22}$$

$$\text{denom}_4 := \left(w_{11,y}^2 + w_{12,y}^2 + w_{21,y}^2 + w_{22,y}^2\right) w_{c,x}^2 -$$
$$2w_{c,x}w_{c,y}\left(w_{11,x}w_{11,y} + w_{12,x}w_{12,y} + w_{21,x}w_{21,y} + w_{22,x}w_{22,y}\right) + \left(w_{11,}^2\right.$$