Covering Radius, Lonely Runner Conjecture, Half-space Depth

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The Covering Radius, Lonely Runner Conjecture

K — convex body in $\mathbb{R}^n$

$$\mu(K) = \min\{\mu > 0 \ : \ \mu K + \mathbb{Z}^n = \mathbb{R}^n\}.$$

Lonely Runner Conjecture – J. Wills '60s, Cusick '73

Given $v_0, v_1, \dots, v_d \in \mathbb{R}$ pairwise distinct.
Then for each $0 \leq i \leq d$ there is a $t \in \mathbb{R}$ such that for all $0 \leq j \leq d$, $i \neq j$, the distance of $t(v_i - v_j)$ to the nearest integer is at least $\frac{1}{d+1}$.

Equivalent Formulation: As a geometric problem: bound the covering radii of certain zonotopes in $\mathbb{R}^n$ with $n = d - 1$.

Accepted: Combinatorica (Q1).

Results

Shifted Lonely Runner Conjecture

Three runners with pairwise distinct velocities, each from her individual starting position plus a Spectator at a fixed position. Then, there exists a time at which all the runners have distance at least $1/4$ from the spectator.

Covering Radius – Efficient Computation

Let $P = \{x \in \mathbb{R}^n : a_i^\intercal x \leq b_i, i \in [m]\}$ be a rational polytope, with $a_i \in \mathbb{Z}^n$ and $b_i \in \mathbb{Z}_{>0}$, for all $i \in [m]$. Then, the algorithm `CoveringRadius`(P) returns the covering radius of P in time

$$\mathcal{O}\left((\|P\|_\infty n)^{2n^2(n+2)} \cdot m^{n+2}\right).$$

The Key: A Geometric Observation

$$N_P := \mathbb{Z}^n \cap ([0,1]^n - \mu_0 P). \tag{1}$$

$$\mu = a_{i_1}^{\mathsf{T}}(x - z_1)/b_{i_1} = \ldots = a_{i_{n+1}}^{\mathsf{T}}(x - z_{n+1})/b_{i_{n+1}} \tag{2}$$

Find a last covered point

Let $P = \{x \in \mathbb{R}^n : a_i^{\mathsf{T}} x \leq b_i, i \in [m]\}$ be a facet description of the polytope P with $b_i > 0$. Assume that $\mu(P) \leq \mu_0$ for some $\mu_0 > 0$, and let N_P be defined by (1).
Then, there are facet normals $a_{i_1}, \ldots, a_{i_{n+1}}$ of P, and not necessarily distinct lattice points $z_1, \ldots, z_{n+1} \in N_P$ such that the system of linear equations (2) in the variables μ and x has a unique solution $(\bar{\mu}, \bar{p})$, and in this solution $\bar{\mu} = \mu(P)$ and $\bar{p}$ is a last-covered point with respect to P.

The Algorithm

Algorithm 1 `CoveringRadius`(P, β_0, μ_0)

$\mu_{\max} := 0$
for $z_1, \ldots, z_{n+1}$ not necessarily distinct points in $\bar{N}_P$ **do**
  for $a_{i_1}, \ldots, a_{i_{n+1}}$ facet normals of P **do**
    if $a_{i_1}/b_{i_1}, \ldots, a_{i_{n+1}}/b_{i_{n+1}}$ are affinely independent **then**
      solve the linear system (2) to obtain (μ, p)
      if $p \in [0,1]^n$ **and** $p \notin \operatorname{int}(()\, \mu P) + \bar{N}_P$ **then** $\{$Is (μ, p) relevant?$\}$
        $\mu_{\max} := \max\{\mu, \mu_{\max}\}$
      end if
    end if
  end for
end for
return $\mu_{\max}$

Half-space Depth

Definition

Given $X \subset \mathbb{R}^d$ finite, $p \in \mathbb{R}^d$.
Take a half-space H that contains the minimum number of points of X among half-spaces containing p.

$$\mathrm{depth}(p, X) = \frac{|X \cap H|}{|X|}.$$

Exact computation: Slow.

Approximation: Finite VC-dimension helps.

Functional problems

John's ellipsoid

For any convex body K in $\mathbb{R}^d$, if the largest volume ellipsoid E contained in K is centered at o, then we have $E \subseteq K \subseteq dE$.

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Functionalization of results on convex sets

Consider *logarithmically concave* functions on $\mathbb{R}^d$, that is, of the form $f(x) = e^{-\varphi}$, where $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{\infty\}$ is a convex function. This yields an extension of the theory of convex sets.

What is the right analogue of an ellipsoid?
Is the Gaussian density (a prime example of a log concave function) a functional ellipsoid?
What is the analogue of John's theorem?

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Motivation: Understanding the volume distribution of high dimensional convex bodies.

Almost accepted: J. of Functional Analysis (Q1).

Simplicial Depth

Definition

Given $X \subset \mathbb{R}^d$ finite, $p \in \mathbb{R}^d$.

$$\mathrm{depth}(p, X) = \frac{|\{Y \subset X : |Y| = d+1, p \in \mathrm{conv}(Y)\}|}{\binom{|X|}{d+1}}.$$

Related phenomenon: Fractional Helly

For every dimension d and $\alpha \in (0,1]$, there is a $\beta \in (0,1]$ such that for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if $\alpha\binom{|\mathcal{F}|}{d+1}$ of the $(d+1)$-tuples of $\mathcal{F}$ intersect, then there is a subfamily $\mathcal{G} \subset \mathcal{F}$ all of whose members intersect. Moreover, $\beta \longrightarrow 1$ as $\alpha \longrightarrow 1$.

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Progress: We almost understand the problem.