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Introduction

Some notations:

- The set of real numbers: $\mathbb{R}$;
- The set of natural numbers: $\mathbb{N} := \{0, 1, 2, \dots\}$;
- The set of positive integers: $\mathbb{N}^+ := \{1, 2, 3, \dots\}$;
- The set of integers: $\mathbb{Z} := \mathbb{N} \cup \{-x \in \mathbb{R} : \ x \in \mathbb{N}\}$;
- The set of rational numbers: $\mathbb{Q} := \left\{ \frac{p}{q} \in \mathbb{R} \ \mid \ p, q \in \mathbb{Z}, \ q \neq 0 \right\}$;
- The set of complex numbers: $\mathbb{C} := \left\{ z = x + iy \in \mathbb{C} \ \middle| \ x, y \in \mathbb{R} \right\}$ (i is the imaginary unit);
- The points of the plane: $\mathbb{R}^2 := \{(x, \ y) \ \mid \ x, y \in \mathbb{R}\}$;
- The points of the space: $\mathbb{R}^3 := \{(x, \ y, \ z) \ \mid \ x, y, z \in \mathbb{R}\}$.

1. Algebraic expressions and radicals I.

1.1. Theoretic review

It is recommended to make a revision of the following topics:

1. Important algebraic identities.
2. Powers (positive integer exponents, negative integer exponents, rational exponents) and operations with exponents.
3. Operations with expressions containing radicals, powers.
4. Factorization of polynomials.

Some important algebraic identities

- $(a+b)^2 = a^2 + 2ab + b^2 \quad (a, b \in \mathbb{R});$
- $(a-b)^2 = a^2 - 2ab + b^2 \quad (a, b \in \mathbb{R});$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad (a, b \in \mathbb{R});$
- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 \quad (a, b \in \mathbb{R});$
- $a^2 - b^2 = (a+b) \cdot (a-b) \quad (a, b \in \mathbb{R});$
- $a^3 + b^3 = (a+b) \cdot (a^2 - ab + b^2) \quad (a, b \in \mathbb{R});$
- $a^3 - b^3 = (a-b) \cdot (a^2 + ab + b^2) \quad (a, b \in \mathbb{R});$
- $a^n - b^n = (a-b) \cdot (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}) \quad (a, b \in \mathbb{R}).$

Polynomials, Roots of polynomials

Let $n \in \mathbb{N}$ be a natural number and the following expression is *a polynomial of degree* n:

$$P(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0, \quad (x \in \mathbb{R})$$

where $a_n \neq 0$, a_{n-1}, ..., a_1, a_0 are given real numbers, the so called *coefficients* of the polynomial P. The number a_n is called the *main coefficient* of P. x denotes the *variable* of P, and stands for any possible real number. In the case when $n = 0$ we are talking about a constant polynomial. These polynomials can be identified with the real numbers. The polynomial 0 is the zero$-$polynomial, for hich we have:

$$P(x) = 0 \quad (x \in \mathbb{R}).$$

The number $\alpha \in \mathbb{R}$ is called the *root* of the polynomial P, if $P(\alpha) = 0$. The first order polynomial

$$x - \alpha$$

is called the root factor corresponding to the root α. Using the identity of decomposing into product $a^n - b^n$ it can be proved, that the number $\alpha \in \mathbb{R}$ is a root of the polynomial P if and only if $x - \alpha$ is a factor in the decomposition of P, which means that there exists a polynomial Q so that:

$$P(x) = (x - \alpha) \cdot Q(x) \qquad (x \in \mathbb{R}).$$

The factorization can be easily made using the Horner-table. Let α be a root of the polynomial P. In this case there is a polynomial Q_1, so that

$$P(x) = (x - \alpha) \cdot Q_1(x) \qquad (x \in \mathbb{R}).$$

If α is also a root of the polynomial Q_1, then there exists a polynomial Q_2, so that:

$$P(x) = (x - \alpha) \cdot (x - \alpha) \cdot Q_2(x) = (x - \alpha)^2 \cdot Q_2(x) \qquad (x \in \mathbb{R}).$$

By continuing this procedure as long as α is still a root of the ”remaining” polynomial, we get that there exists a number $m \in \mathbb{N}^+$ and a polynomial Q_m with the following properties:

$$P(x) = (x - \alpha)^m \cdot Q_m(x) \qquad (x \in \mathbb{R}),$$

and α is not a root of Q_m, which means that $Q_m(\alpha) \neq 0$.
We call by definition m the multiplicity of α (in the polynomial P).

Using the upper method of factorizing P it can be proved, that a polynomial of degree n has at most n roots.

1.1.1. Checking questions to the theory and its use

1. Evaluate: $(a - 2b + c)^2$, if $a, b, c \in \mathbb{R}$.
2. Make the factorization of the following expression, if $x, y, z \in \mathbb{R}$:
$$4x^2y^2 - (x^2 + y^2 - z^2)^2.$$
3. Give the most simple algebraic form of the following polynomial:
$$P(x) := (2x - 1)^3 - 2 \cdot (2 + x)^3 + x + 5 \quad (x \in \mathbb{R}).$$
4. Decompose the following polynomial into factors:
$$P(x) := 3x^3 + 3x^2 - 6x \quad (x \in \mathbb{R}).$$

5. Simplify the following algebraic fraction:

$$E(x) := \frac{x^3-1}{1-x^5} \quad (1 \neq x \in \mathbb{R}).$$

6. Simplify the following algebraic fraction:

$$E(x) := \frac{x^4+x^2+1}{x^6-1} \quad (\pm 1 \neq x \in \mathbb{R}).$$

7. Simplify the following algebraic fraction:

$$E(x) := \frac{x^2+x+1}{x^4+x^2+1} \quad (x \in \mathbb{R}).$$

8. Evaluate the indicated operations and simplify the following expression:

$$\left(\frac{y^2}{x^3-xy^2}+\frac{1}{x+y}\right) : \left(\frac{x-y}{x^2+xy}-\frac{x}{y^2+xy}\right) \quad (x; y \in \mathbb{R}; x \neq 0; |x| \neq |y|).$$

9. Rationalize the denominator of the following fractions:

   (a) $\dfrac{2}{\sqrt{3-\sqrt{2}}-1}$;

   (b) $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{\sqrt{2}}}}$.

10. Evaluate the exact value of the following expressions:

   (a) $\left(\sqrt{9+6\sqrt{2}}-\sqrt{9-6\sqrt{2}}\right)^2$;

   (b) $\dfrac{1}{\sqrt{3-2\sqrt{2}}}-\dfrac{1}{\sqrt{3+2\sqrt{2}}}$.

11. What is the exact value of:

$$\left(\frac{1}{\sqrt{5}-2}\right)^3-\left(\frac{1}{\sqrt{5}+2}\right)^3?$$

12. Evaluate:

$$\frac{\sqrt{a}+\sqrt{b}-1}{a+\sqrt{ab}}+\frac{\sqrt{a}-\sqrt{b}}{2\sqrt{ab}}\cdot\left(\frac{\sqrt{b}}{a-\sqrt{ab}}+\frac{\sqrt{b}}{a+\sqrt{ab}}\right) \quad (a; b > 0; ab \neq 0; a \neq b).$$

13. Evaluate:

$$\frac{\left(a^{1/2}\cdot b^{2/3}\right)^{-3/4}\cdot\left(a^{1/3}\cdot b^{1/4}\right)^2}{\left(a^{1/12}\right)^{-1/2}} \quad (a > 0; b \geq 0).$$

14. Consider the following function:

$$f(x) := \sqrt{x + 2\sqrt{x} + 1} + \sqrt{x - 2\sqrt{x} + 1} \quad (x \in [0; +\infty))\,.$$

What is the most simple form of $f(x)$?

15. Let $x, y, z \in \mathbb{R} \setminus \{0\}$ be real numbers, for which:

$$x + y + z = 1 \ \wedge \ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

Evaluate:

$$x^2 + y^2 + z^2\,.$$

1.2. Exercises

1.2.1. Exercises for class work

Algebraic computations, identities

1. Prove the following identity for all $a, b \in \mathbb{R}$:

$$a^2 + ab + b^2 = 3 \cdot \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2.$$

Evaluate $a^3 - b^3$, if $a - b = 2$ and $a + b = \sqrt{5}$.

2. Let $x > 0$ be a positive number for which: $x^2 + \dfrac{1}{x^2} = 7$.

Prove that $x^5 + \dfrac{1}{x^5}$ is an integer.

3. Prove that:

(a) $\dfrac{1}{(a+b)^2}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) + \dfrac{2}{ab(a+b)^2} = \dfrac{1}{a^2b^2}$;

(b) $\dfrac{a}{a^3 + a^2b + ab^2 + b^3} + \dfrac{b}{a^3 - a^2b + ab^2 - b^3} + \dfrac{1}{a^2 - b^2} - \dfrac{1}{a^2 + b^2} - \dfrac{a^2 + 3b^2}{a^4 - b^4} = 0$;

(c) $\dfrac{1}{a(a-b)(c-a)} + \dfrac{1}{b(a-b)(b-c)} + \dfrac{1}{c(c-a)(b-c)} = -\dfrac{1}{abc}$;

(d) $\dfrac{a^2 - bc}{(a+b)(a+c)} + \dfrac{b^2 - ac}{(a+b)(b+c)} + \dfrac{c^2 - ab}{(a+c)(b+c)} = 0.$

4. Prove that, if

(a) $a, b, c \in \mathbb{R}$ and $a+b+c=0$, then $a^3+a^2c-abc+b^2c+b^3=0$;

(b) $a, b, c \in \mathbb{R}$ and $a+b+c=0$, then $a^3+b^3+c^3=3abc$;

(c) $a, b, c \in \mathbb{R}$ and $a^2+b^2+c^2=ab+ac+bc$, then $a=b=c$.

5. Prove that, if the positive real numbers $x, y, z > 0$ satisfy the following conditions:

$$x+2y+3z=6 \quad \wedge \quad \frac{6}{x}+\frac{3}{y}+\frac{2}{z}=1,$$

then

$$(x-6)\cdot(y-3)\cdot(z-2)=0.$$

6. Simplify the following algebraic fraction, where x, y are real numbers, but $x \neq y$:

$$E(x,y)=\frac{x^3-x-y^3+y+xy^2-x^2y}{x^3+x-y^3-y+xy^2-x^2y}.$$

Prove that if we substitute into variables x, y the following expressions, then the result does not depend on the parameter z:

$$x=\frac{k(1-z^2)}{1+z^2}; \quad y=\frac{2kz}{1+z^2} \qquad (k, z \in \mathbb{R}).$$

7. Decompose into product the following expression:

$$(a^2+b^2-c^2)^2-(a^2-b^2+c^2)^2 \qquad (a, b, c \in \mathbb{R}).$$

8. Evaluate the following sum, where in the last number there are n digits of 1

$(1 \leq n \in \mathbb{N})$:

$$1+11+111+\cdots+\underbrace{1\cdots1}_{n \text{ digits}}.$$

9. Consider the following functions:

$$f(x):=\frac{1-x}{1+x} \quad (x \in \mathbb{R}\setminus\{-1\}); \quad g(x):=\frac{1+x}{1-x} \quad (x \in \mathbb{R}\setminus\{1\}).$$

Prove that for all $x \in \mathbb{R}\setminus\{-1;0;1\}$ we have:

$$f(g(x))\cdot g(f(x))+1=0.$$

10. Find a function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ which makes true the following equation:

$$\frac{x^n + 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x^{n-k} + f(x) \quad (1 \le k \le n; 1 \le k, n \in \mathbb{N})$$

for all $x \in \mathbb{R} \setminus \{1\}$.

Expressions and computations with radicals, roots

11. Prove that:

(a) $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) \qquad (a, b \ge 0)$;

(b) $a + b = \left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}\right) \qquad (a, b \in \mathbb{R})$;

(c) $a - b = \left(\sqrt[3]{a} - \sqrt[3]{b}\right)\left(\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}\right) \qquad (a, b \in \mathbb{R})$.

12. Evaluate and simplify the following expressions:

(a) $\dfrac{\sqrt{a+b} - \sqrt{a-b}}{\sqrt{a+b} + \sqrt{a-b}} \qquad (a, b \in \mathbb{R},\ 0 < b < a)$;

(b) $\dfrac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a} + \sqrt{b}} \qquad (a, b \in \mathbb{R},\ 0 < b, a)$;

(c) $\left(\sqrt{x} - \dfrac{\sqrt{xy} + y}{\sqrt{x} + \sqrt{y}}\right) \cdot \left(\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}} + \dfrac{\sqrt{y}}{\sqrt{x} - \sqrt{y}} + \dfrac{2\sqrt{xy}}{x - y}\right) \quad (0 < x \ne y)$.

13. Give the most simple form of the following expression $(0 < x, y;\ \ x \ne y)$:

$$E(x, y) := \left(\frac{x^{-1/6} - \dfrac{5}{\sqrt[6]{y}}}{\dfrac{1}{x^{1/3}} - y^{-1/3}} - 5 \cdot \frac{x^{-1/6} - y^{-1/6}}{x^{-1/3} - \sqrt[3]{y^{-1}}} \right)^{-1} \cdot \frac{6\sqrt[6]{x}}{\sqrt[3]{x} - \sqrt[3]{y}}.$$

14. Evaluate and give the most simple form of the following expression, and evaluate its value for $x = 0,5$:

$$E(x) := \left(\frac{\sqrt{1+x}}{\sqrt{1+x} - \sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2} - 1 + x} \right) \cdot \left(\sqrt{x^{-2} - 1} - \frac{1}{x} \right) \qquad (0 < x < 1)\ .$$

15. Give the most simple form of the following number:

$$\sqrt{7-\sqrt{48}}+\sqrt{5-\sqrt{24}}+\sqrt{3-\sqrt{8}}.$$

16. Evaluate the following *"telescopic"* sums:

$$\text{a)}\quad \sum_{k=1}^{n}\frac{1}{k^2+k}\quad (1\le n\in\mathbb{N});$$

$$\text{b)}\quad \sum_{k=1}^{n}\frac{1}{\sqrt{k-1}+\sqrt{k}}\quad (1\le n\in\mathbb{N}).$$

17. Consider the following function:

$$f(x):=\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}\quad (x\in[1;+\infty))\,.$$

Give the most simple form of f and sketch its graph.

Polynomials

18. Decompose into factors the following polynomials:

(a) $P(x):=x^3+8\quad (x\in\mathbb{R})$;

(b) $Q(x):=x^5+x^4+x^3+x^2+x+1\quad (x\in\mathbb{R})$.

19. Prove that the given number x_0 is a root of the polynomial P and divide P with the root factor $(x-x_0)$:

(a) $x_0=2,\quad P(x)=3x^2-7x+2$;

(b) $x_0=3,\quad P(x)=2x^3-4x^2-18$;

(c) $x_0=-1,\quad P(x)=2x^4-5x^3-6x^2+3x+2$.

20. For what value of $k\in\mathbb{R}$ can we multiply out

(a) $(x+3)$ from $(2x^2+x+k)\quad (x\in\mathbb{R})$;

(b) $(x-3)$ from $(4x^2-6x+k)\quad (x\in\mathbb{R})$?

Give the product form of P with those factors.

21. Let x_1, x_2 be the real roots of $x^2 + px + 1$ and x_3, x_4 the real roots of $x^2 + qx + 1$. Evaluate the following product using p and q:

$$(x_1 - x_3) \cdot (x_2 - x_3) \cdot (x_1 + x_4) \cdot (x_2 + x_4).$$

22. Find $a, b, c \in \mathbb{R}$ so that the polynomial:

$$P(x) = x^4 + x^3 + ax^2 + bx + c \quad (x \in \mathbb{R})$$

divided by $(x - 1)$, $(x - 2)$ and $(x - 3)$ gives the remainders $1, 2$ respectively 3.

1.2.2. Homework and exercises to practice

Algebraic computations, identities

1. Evaluate and give the most simple form of the following expression:

$$\frac{a}{a^2 - 2ab + b^2} - \frac{a}{a^2 - b^2} + \frac{1}{a + b} \quad (a, b \in \mathbb{R},\ |a| \neq |b|).$$

2. Prove the following identity:

$$\left(\frac{a}{a + 2b} - \frac{a + 2b}{2b}\right)\left(\frac{a}{a - 2b} - 1 + \frac{8b^3}{8b^3 - a^3}\right) = \frac{a}{2b - a}$$

$(a, b \in \mathbb{R},\ |a| \neq 2|b|,\ b \neq 0)$.

3. Prove that for all the real numbers $a, b, c, x, y \in \mathbb{R}$ we have:

(a) $a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 4abc = (a + b)(b + c)(c + a)$;

(b) $(a^2 + b^2)(x^2 + y^2) - (ax + by)^2 = (ay - bx)^2$;

(c) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

4. Prove the following identities for all $a, b, c \in \mathbb{R}$:

(a) $(a + b)^3 + (b + c)^3 + (c + a)^3 - 3(a + b)(b + c)(a + c) = 2(a^3 + b^3 + c^3 - 3abc)$;

(b) $(a - b)^3 + (b - c)^3 + (c - a)^3 - 3(a - b)(b - c)(c - a) = 0$;

(c) $(a^2 - bc)^3 + (b^2 - ac)^3 + (c^2 - ab)^3 - 3(a^2 - bc)(b^2 - ac)(c^2 - ab) =$
$= (a^3 + b^3 + c^3 - 3abc)^2$;

(d) $(a + b - c)^3 + (b + c - a)^3 + (c + a - b)^3 - 3(a + b - c)(b + c - a)(c + a - b) =$
$= 4(a^3 + b^3 + c^3 - 3abc)$.

5. Prove that if $a, b, c > 0$ and $abc = 1$, then

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} = 1.$$

6. Consider the real numbers a, b, c so that:

$$a+b+c = 1 \quad \wedge \quad \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} = 1.$$

Evaluate the value of the following expression:

$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ac}.$$

7. Prove that if $x, y, z \in \mathbb{R}$ and

$$x^3 + y^3 + z^3 = x^2 + y^2 + z^2 = x + y + z = 1,$$

then $xyz = 0$.

8. What is the value of

$$a^3 + b^3 + 3(a^3b + ab^3) + 6(a^3b^2 + a^2b^3),$$

if $a, b \in \mathbb{R}$ and $a + b = 1$.

9. Consider the integers x, y so that $x - y = 2$. Prove that $x^3 - y^3$ can be written as the sum of three squared integers.

10. Evaluate and simplify the following expression:

$$\left(\frac{4}{3x} - \frac{1}{x-1}\right) : \left(1 - \frac{3(x-2)}{2(x-1)}\right) \quad (x \in \mathbb{R},\ x \neq 0; 1).$$

11. Evaluate the following expressions at the given values of the variables:

(a) $\dfrac{(a+1)(a^8+a^4+1)}{(a^4-a^2+1)(a^2+a+1)}$, ha $a = 10$;

(b) $\left(\dfrac{8+b^3}{x^2-y^2} : \dfrac{4-2b+b^2}{x-y}\right)\left(x + \dfrac{xy+y^2}{x+y}\right)$, ha $b = 8,\ x = 997,5,\ y = -0,75$.

12. Consider the following functions:

$$f(x) := 2x + 3 \quad (x \in \mathbb{R}); \quad g(x) := 4x + 9 \quad (x \in \mathbb{R}).$$

Prove that we have:

$$f(g(x)) = g(f(x)) \quad (x \in \mathbb{R}).$$

13. Assume that $x + \dfrac{1}{x} = a \in \mathbb{R}$. Express $x^4 + \dfrac{1}{x^4}$ with a.

14. Find a function $f : \mathbb{R} \setminus \{0; 1\} \to \mathbb{R}$ so that the following equation holds for all $x \in \mathbb{R} \setminus \{0; 1\}$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^{n+1} \cdot f(x) \quad (1 \leq n \in \mathbb{N}).$$

How can we define the value $f(0)$ so that, the equation above to be true for $x = 0$ as well? How much is the value $\dfrac{1}{f(2018)}$?

Expressions and evaluations with radicals, roots

15. Prove that:

(a) $\sqrt{a} - \sqrt{b} = (\sqrt[4]{a} - \sqrt[4]{b})(\sqrt[4]{a} + \sqrt[4]{b}) \quad (a, b \geq 0)$;

(b) $a\sqrt{a} - b\sqrt{b} = (\sqrt{a})^3 - (\sqrt{b})^3 = (\sqrt{a} - \sqrt{b})(a + \sqrt{ab} + b) \quad (a, b \geq 0)$;

(c) $\sqrt{a^3} + \sqrt{b^3} = (\sqrt{a} + \sqrt{b})\left(a - \sqrt{ab} + b\right) \quad (a, b \geq 0)$.

16. Give the most simple form of the following expression:

$$\frac{a-b}{a+b} \cdot \sqrt{\frac{a^2 + ab}{a^2 - 2ab + b^2}} \quad (a, b \in \mathbb{R},\ 0 < b < a).$$

17. Evaluate the value of the following expression:

$$\frac{2\sqrt{xy} + 4\sqrt{y} - 3\sqrt{x} - 6}{2 - 2y} : \left(\frac{4y + 19 - 2\sqrt{y}}{2 + 2\sqrt{y}} - 5\right),$$

if $x = 16,\ y = 9$.

18. Prove that:

(a) $\sqrt{7 + 2\sqrt{6}} \cdot \sqrt{7 - 2\sqrt{6}}$;

(b) $\sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2}$;

(c) $\sqrt{7 + 4\sqrt{3}} + \sqrt{7 - 4\sqrt{3}}$;

(d) $\sqrt[3]{1 - 27\sqrt[3]{26} + 9\sqrt[3]{26^2}} + \sqrt[3]{26}$

are integers.

19. Evaluate the following *"telescopic"* sums:

$$\text{a)}\quad \sum_{k=1}^{n} \frac{2k+1}{k^2\cdot(k+1)^2} \quad (1\le n\in\mathbb{N});$$

$$\text{b)}\quad \sum_{k=1}^{n} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}} \quad (1\le n\in\mathbb{N}).$$

20. Give the most simple form of the following expression:

$$E(x,y) := \left(\frac{x^{1/2}+y^{1/2}}{(x+y)^{1/2}} - \frac{(x+y)^{1/2}}{x^{1/2}+\left(\dfrac{1}{y}\right)^{-1/2}}\right)^{-2} - \frac{x+y}{2\sqrt{xy}}; \qquad (0<x,y).$$

21. Give the most simple form of the following expression:

$$E(a,b) := \frac{(ab)^{3/2}-(a-b)\cdot\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right)}{ab\left(\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}+\sqrt{ab}\right)+(a-b)\left(\sqrt{\dfrac{b}{a}}-\sqrt{\dfrac{a}{b}}-\sqrt{ab}\right)}.$$

Polynomials

22. Give the best possible product forms for the following polynomials:

(a) $4x^2-9b^2$;
(b) y^3+1;
(c) $8a^3-27$;
(d) $27a^3+8$;
(e) $8a^3+b^6$;
(f) $27a^6x^{12}-64b^9y^{15}$;
(g) $x^3+18x^2+108x+216$.

23. Prove that the given number x_0 is a root of the given polynomial P and multiply out the root factor $(x-x_0)$ from P:

(a) $x_0=1,\quad P(x)=5x^3-2x^2+7x-10$;

(b) $x_0 = -2, \quad P(x) = 3x^3 + 10x^2 + 8x.$

24. For what values of $k \in \mathbb{R}$ can we multiply out:

(a) $(x-4)$ from $(x^3 - 4x + 2k)$;

(b) $(x+1)$ from$(x^4 - 3x^3 + 5x^2 + 7x - 3k)$?

Give the product form of f by multiplying out the $(x - x_0)$ factor.

25. Find all the integer roots of the following polynomials:

(a) $x^4 - 2x^3 - 8x^2 + 13x - 4$;

(b) $x^4 - 2x^3 - 8x^2 + 13x - 6$;

(c) $x^3 - 6x^2 + 15x - 14$;

(d) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$;

(e) $x^5 - 7x^3 - 12x^2 + 6x + 36$.

26. Consider the polynomial

$$P(x) := x^2 + ax + b \quad (x \in \mathbb{R}),$$

where a and b are real parameters. Evaluate the following expression:

$$P(-P(x)) - P(P(-x)) \quad (x \in \mathbb{R}).$$

2. Quadratic equations, inequalities

2.1. Theoretic review

It is recommended to go through the following topics:

1. Quadratic equations, quadratic formula to evaluate the roots, quadratic inequalities.
2. The Viéte formulas.
3. Decomposition into factors, complete to square.
4. Quadratic functions, polynomials and their properties.
5. Quadratic functions, their graphs, parabolas and their properties.
6. Minimal and maximal values of quadratic functions.
7. Extremum exercises.

2.1.1. Checking questions to the theory and its use

1. Let $a, b, c \in \mathbb{R}; a \neq 0$ be fixed real numbers. What are the roots of the quadratic equation
$$ax^2 + bx + c = 0$$
and discuss their nature depending on the sign of the discriminant.
2. Let $a, b, c \in \mathbb{R}; b \neq 0$ be fixed real numbers. Write down the Viéte formulas between the roots and the coefficients of the equation
$$bx^2 + ax + c = 0.$$
3. Consider the equation $2x^2 - 3x - 8 = 0$. Evaluate the expression:
$$\frac{x_1}{x_2} + \frac{x_2}{x_1},$$
if x_1, x_2 are the roots of the given equation.
4. Which is the quadratic equation with the dominant coefficient 1 whose roots are $\sqrt{2} - 1$ and $\sqrt{2} + 1$. Give all the coefficients of this equation.
5. Let $p \in \mathbb{R}$ be a real parameter. For what values of p has the following equation two different real roots:
$$px^2 - x + 1 = 0\,?$$

6. Let $p \in \mathbb{R}$ be a real parameter. For what values of p is the following inequality true for all $x \in \mathbb{R}$:
$$px^2 - (p+2)x + 3 > 0\,?$$

7. Let $p \in \mathbb{R}$ be a real parameter. Find all the values of p so that the following equation has two different real roots:
$$\frac{p}{x^2} - \frac{2}{x} + 1 = 0\,?$$
For what p has this equation only one real root?

8. Solve the following inequality on the set of the real numbers:
$$\frac{x^2 - 2x}{x^2 - x + 1} < 1.$$

9. Consider the function
$$f(x) := x^2 - 4x + 7 \quad (x \in \mathbb{R}).$$
Give the "complete square" form of f, draw the graph of f and give the smallest and the biggest values of f (if they exist). Where does the function f take its extremal values? What is the intersection point of the graph of f and the axis y?

10. What real numbers x, y satisfy the following equation:
$$x^2 - 3xy + y^2 = 0\,?$$
Where are these $(x; y)$ points on the plane?

11. Evaluate the smallest and biggest value of the following function:
$$f(x) := 2 - \frac{1}{x^2 - 2x + 2} \quad (x \in [-3; 2]).$$
Where does the f take them?

12. Prove that for all $a, b \geq 0$ non-negative real numbers the following inequality holds:
$$\frac{a+b}{2} \geq \sqrt{ab}.$$
When do we have equality here?

13. Prove that for all real numbers $x \in \mathbb{R}$ the following inequality is true:
$$\frac{|x|}{x^2 + 4} \leq \frac{1}{4}.$$
When do we have equality here?

14. Find the extremal points (place and value) of the following f:
$$f(x) := -2x^2 + x - 1 \quad (x \in D := \{x \in \mathbb{R} : |x - 1/2| \leq 1/2\}).$$

15. Consider the real parameters $a, b, c \in \mathbb{R}; a \neq 0$ and the quadratic polynomial

$$P(x) := ax^2 + bx + c \quad (x \in \mathbb{R}).$$

Give the factorization of f, if x_1 and x_2 denote the real roots of the equation

$$f(x) = 0.$$

In what cases is f a complete square? Give this complete square form too.

2.2. Exercises

2.2.1. Exercises for class work

1. Give the complete square form of the following polynomials and using this form solve the equation $P(x) = 0$:

$$P(x) = x^2 - 6x + 3; \qquad P(x) = 2x^2 + 7x - 1.$$

2. Using the *Viète* formulas evaluate for the roots of the equations above the following expressions:

(a) the sum of the roots;

(b) the product of the roots;

(c) the sum of the squared roots;

(d) the absolute value of the difference of the roots;

(e) the sum of the reciprocals of the roots;

(f) the sum of the cubes of the roots.

3. Solve the following inequalities ($x \in \mathbb{R}$):

(a) $x^2 - 5x + 6 > 0$;

(b) $\dfrac{3x^2 + 7x - 4}{x^2 + 2x - 3} < 2$;

(c) $\dfrac{x-1}{x+1} > \dfrac{3x+4}{1-2x}$;

(d) $\dfrac{x+2}{x+1} + \dfrac{3x-2}{1-2x} \leq 0.$

4. Find all the values of the real parameter $p \in \mathbb{R}$ so that:

(a) the inequality $x^2 + 6x + p > 0$ is true for all $x \in \mathbb{R}$!

(b) the inequality $x^2 - px > \dfrac{2}{p}$ holds for all $x \in \mathbb{R}$!

(c) the inequality $(p^2 - 1)x^2 + 2(p - 1)x + 1 > 0$ is true for all $x \in \mathbb{R}$!

(d) the inequality $\dfrac{x^2 - px + 1}{x^2 + x + 1} < 3$ is true for all $x \in \mathbb{R}$!

5. For some value of the real parameter $p \in \mathbb{R}$ the squared sum of the roots of the equation

$$2x^2 - 3(p - 1)x + 1 - p^2 = 0$$

is $\dfrac{5}{4}$. What is p?

6. Find the parameter $p \in \mathbb{R}$ so that the roots of the quadratic equation

$$(1 - p)x^2 + 2px = p + 3$$

be different and both positive.

7. Let $p \in \mathbb{R}$ and consider the equation $x^2 - (p - 2)x + p - 3 = 0$. Find those values of the parameter p for which the sum of the square of the real roots of this equation is minimal.

8. For what coefficients $p, q \in \mathbb{R}$ will be p and q both roots of the equation

$$x^2 + px + q = 0?$$

9. Prove that for all $x \in \mathbb{R}$ we have:

$$\frac{7 - \sqrt{52}}{3} \leq \frac{x + 3}{x^2 - x + 1} \leq \frac{7 + \sqrt{52}}{3}.$$

10. Find the minimal and maximal values of the following function:

$$f(x) = \frac{x^2 - 3x + 2}{x^2 + 4x + 5} \quad (x \in \mathbb{R}).$$

11. Suppose that a, b, c are three different consequtive terms of a geometric sequence. Prove the following inequalities:

$$\frac{1}{3} \leq \frac{ax^2 + bx + c}{ax^2 - bx + c} \leq 3 \quad (x \in \mathbb{R}).$$

12. Prove that:

(a) $\dfrac{2}{1/a+1/b} \le \sqrt{ab} \le \dfrac{a+b}{2} \le \sqrt{\dfrac{a^2+b^2}{2}} \quad (a,b \in (0;+\infty))$;

(b) $a^2+b^2+c^2 \ge ab+ac+bc \quad (a,b,c \in \mathbb{R})$;

(c) $\left|a+\dfrac{1}{a}\right| \ge 2 \quad (a \ne 0)$.

When do we have equalities in the upper inequalities?

13. Prove that for all $a, b, c \in \mathbb{R}$ we have:

$$(a^2+1)(b^2+1)(c^2+1) \ge 8abc.$$

When do we have equality here?

14. Prove the following inequalities:

(a) $\left|\dfrac{1}{a-b}\right| < \dfrac{2}{|a|} \quad (a,b \in \mathbb{R},\ 2|b| < |a|)$;

(b) $\left|\dfrac{a}{b}+\dfrac{b}{a}\right| \ge 2 \quad (a,b \in \mathbb{R}\backslash\{0\})$;

(c) $\dfrac{x^2+2}{\sqrt{x^2+1}} \ge 2 \quad (x \in \mathbb{R})$;

(d) $\dfrac{x^2}{1+x^4} \le \dfrac{1}{2} \quad (x \in \mathbb{R})$;

(e) $a^2+b^2-ab-a-b+1 \ge 0 \quad (a,b \in \mathbb{R})$;

(f) $2 < \left(\dfrac{a+2b}{a+b}\right)^2 \quad (a,b \in (0,+\infty),\ a^2 < 2b^2)$.

15. Prove that for all $a,\ b,\ x,\ y \in \mathbb{R}$ we have:

(a) $|ax+by| \le \sqrt{a^2+b^2} \cdot \sqrt{x^2+y^2}$ *(Cauchy–Bunyakovszkij inequality)*;

(b) $\sqrt{(x+a)^2+(y+b)^2} \le \sqrt{x^2+y^2} + \sqrt{a^2+b^2}$ *(Minkowski inequality)*;

We have equalities here, if there exists a number $\lambda > 0$, so that

$$(x = \lambda a \text{ and } y = \lambda b) \quad \text{or} \quad (a = \lambda x \text{ and } b = \lambda y)\ .$$

What is the geometric meaning of these inequalities?

2.2.2. Homework and exercises to practice

1. Give the complete square form of the following polynomials and using this form solve the equation $P(x) = 0$:

$$P(x) = x^2 + 10x + 26; \qquad P(x) = -x^2 + 2x + 3; \qquad P(x) = -3x^2 + 8x + 5.$$

2. Using the *Viète* formulas evaluate for the roots of the equations above the following expressions:

(a) the sum of the roots;
(b) the product of the roots;
(c) the sum of the squared roots;
(d) the absolute value of the difference of the roots;
(e) the sum of the reciprocals of the roots.
(f) the sum of the cubes of the roots.

3. Solve the following inequalities, where $x \in \mathbb{R}$:

(a) $\dfrac{2x^2 + 5x - 18}{x - 2} \leq 0;$
(b) $\dfrac{x - 9}{\sqrt{x} - 3} \geq 0\ .$

4. For what coefficients $p, q \in \mathbb{R}$ will the equation $x^2 + px + q = 0$

(a) have a root so that its reciprocal is also a root?
(b) have the property that all the reciprocals of all of its roots are also roots?
(c) have the property that all the squares of all of its roots are also roots?
(d) have the property that all the opposites of all of its roots are also roots?

5. Solve the following equations (where $p \in \mathbb{R}$ is a real parameter):

(a) $x(x + 3) + p(p - 3) = 2(px - 1);$
(b) $\dfrac{x(x - p)}{x + p} - x + p = \dfrac{10x}{x + p} - 10.$

6. For what values of the parameter $m \in \mathbb{R}$ will the equation

$$(m - 1)x^2 - 2mx + m - 2 = 0$$

have two different real roots?

7. Find the values of the parameter $m \in \mathbb{R}$ so that both roots of the following equation to be positive

$$x^2 + 2(m-3)x + m^2 - 4 = 0.$$

8. For what values of the parameter $p \in \mathbb{R}$ has the equation

$$(1-p)x^2 - 4px + 4(1-p) = 0$$

at most one real root?

9. Find $q \in \mathbb{R}$ so that the equation $x^2 - 4x + q = 0$

(a) has the property that it has a root whose tripled value is also a root;

(b) has exactly one root.

10. For what numbers $k \in \mathbb{R}$ will the following euations have common root(s):

$$x^2 + kx + 1 = 0 \quad \text{és az} \quad x^2 + x + k = 0?$$

11. Solve the following equation on the set of the real numbers:

$$(2x^2 + 7x - 8) \cdot (2x^2 + 7x - 3) - 6 = 0.$$

12. Consider the parameters $a, b \in \mathbb{R}$ and we know that $x_1 = -2$ is one root of the equation

$$x^3 + ax^2 + x + b = 0.$$

It also has the property that it has a root whose reciprocal is also root. Find the parameters a, b.

13. Suppose that the roots x_1, x_2 of a quadratic equation satisfy the following properties:

$$4x_1x_2 - 5x_1 - 5x_2 + 4 = 0; \quad (x_1 - 1) \cdot (x_2 - 1) = \frac{1}{1+a},$$

where $a \in \mathbb{R} \setminus \{-1\}$ is a real parameter.

a) Give the quadratic equations whose roots are the upper x_1 and x_2.

b) Find the values of a so that:

$$x_1^2 + x_2^2 = 11.$$

3. Algebraic expressions and radicals II.

3.1. Theoretic review

Absolute value, its properties and the triangle inequalities

Definition : Let $x \in \mathbb{R}$ be an arbitrary fixed real number. Its absolute value $|x|$ is defined by:

$$|x| := \begin{cases} -x & \text{, if } x < 0; \\ x & \text{, if } x \geq 0. \end{cases}$$

It is obvious that $|x|$ is the distance of x from 0 on the real coordinate line.

If $x, y \in \mathbb{R}$ are real numbers, then the nonnegative number $|x-y|$ measures the geometric distance between x and y on the real coordinate line.

It is easy to prove that the following properties are true:

$$\forall\, x, y \in \mathbb{R}: \;\; |x \cdot y| = |x| \cdot |y|,$$

and

$$\forall\, x, y \in \mathbb{R}, y \neq 0: \;\; \left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

We cannot state the same properties for the absolute value of a sum or a difference, but the following important inequalities are true:

Theorem: (*Triangle inequalities*) For all real numbers $x, y \in \mathbb{R}$ we have:

1. $|x+y| \leq |x| + |y|$;
2. $|x-y| \geq \Big||x| - |y|\Big|$.

3.1.1. Checking questions to the theory and its use

1. Define the absolute value of a real number x.
2. Depending on the values of $x \in \mathbb{R}$ how can we eliminate the absolute value in $|x^2 - 1|$?
3. Write down the *triangle inequalities*.
4. Give all the points $(x; y)$ in the plane which satisfy the inequality: $|y - |x|| < 1$.
5. Solve the equation: $|x+2| = x - 1$.
6. For what real numbers x holds the inequality $||x-1| - 2| > 1$?

7. Simplify the following expression:

$$\frac{x^3-1}{x^4-1}.$$

8. Solve the following equation:

$$\frac{1}{\sqrt{x}-1}-\frac{1}{\sqrt{x}+1}=\frac{3}{x}.$$

9. Solve the following inequality:

$$\sqrt{x-4}+\sqrt{3-x}<x+\sqrt{x}-1.$$

10. For what real numbers x is true that:

$$|2-\sqrt{1-x}|<1?$$

11. For what real numbers is true that:

$$\sqrt{x^2}=x+1?$$

12. Give two different real numbers x, t (if they exist), so that:

$$(x-2)^2+|x-1|=(t-2)^2+|t-1|.$$

13. We know that x satisfies the condition $|x+1|<1/2$. What values can $|x-1|$ take?
14. Find all the real numbers x for which:

$$|x|<2 \text{ and } |1-x|>1\ .$$

15. The real number x satisfies the condition $|x-3|<2$. Give an upper estimate for $|x^2-9|$.

3.2. Exercises

3.2.1. Exercises for class work

Algebraic computations and simplifications

1. Give the most simple forms of the following expressions:

(a) $\dfrac{3x^2+5x-2}{x^2+3x+2}$;

(b) $\dfrac{x^4+5x^2+4}{x^4-16}$;

(c) $\dfrac{2x^2-13x-7}{8x^3+1}$;

(d) $\dfrac{(x^2-1)^2-(x^2+1)^2}{x^5+x^3}$;

(e) $\dfrac{2}{x^2-1}-\dfrac{3}{x^3-1}$.

2. Make the rationalization of these fractions and then simplify them:

(a) $\dfrac{\sqrt{x^2+1}-\sqrt{2}}{x^3-1}$;

(b) $\dfrac{x^2+x-6}{\sqrt{\sqrt{x}-\sqrt{2}+1}-1}$;

(c) $\dfrac{x^2-64}{\sqrt[3]{x}-2}$;

(d) $\dfrac{x^2-3x-4}{\sqrt{\sqrt{x}-1}-1}\quad (x>4)$.

Equations and inequalities containing absolute values

3. For what numbers $x \in \mathbb{R}$ are true the following equations and inequalities:

(a) $|2x-7|+|2x+7|=14$;

(b) $|2x-7|+|2x+7|=x+15$;

(c) $|x^2-4x-5|+|x-2|=7$;

(d) $|x^2+3x|=|2x-6|$;

(e) $|x^2-9|+|x^2-4|=5$;

(f) $|x-2|<3$;

(g) $|2x-1|<|x-1|$;

(h) $|x(1-x)|<\dfrac{1}{4}$;

(i) $\dfrac{1+|x-2|}{|x-3|}\leq\dfrac{1}{2}$.

4. The number x satisfies the condition $|x+2|<1$. Between what bounds can change $|x+1|$?

5. The number x satisfies the condition $|x-2|<2$. Give an upper estimate for $|x^2-4|$.

Equations and inequalities containing radicals, roots

6. Solve the following equations and inequalities:

(a) $\sqrt{x+1}-\sqrt{9-x}=\sqrt{2x-12}\,;$

(b) $\sqrt{x^2+5x+1}=2x-1\,;$

(c) $x-1=\sqrt{1-x\sqrt{16+x^2}}\,;$

(d) $\sqrt{6x^2+8x-8}-\sqrt{3x-2}=0\,;$

(e) $\sqrt{x^2-p}+2\cdot\sqrt{x^2-1}=x \quad (p\in\mathbb{R})\,;$

(f) $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\cdot\sqrt{\frac{x}{x+\sqrt{x}}}\,;$

(g) $\frac{x\sqrt[3]{x}-1}{\sqrt[3]{x^2}-1}-\frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}-1}=12\,;$

(h) $\sqrt{|1-x^2|}=\frac{x}{2}+1\,;$

(i) $\sqrt[3]{4x-1}+\sqrt[3]{4-x}=-\sqrt[3]{3};$

(j) $\sqrt{3x+13}\le x+1\,;$

(k) $\sqrt{x^2+4x}>2-x\,;$

(l) $\sqrt{x^2-1}<5-x\,;$

(m) $\frac{4x^2}{(1-\sqrt{1+2x})^2}<2x+9\,;$

(n) $\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}\,;$

(o) $\sqrt{2x+1}-\sqrt{x-8}>3\,;$

(p) $\frac{1+\sqrt{x}}{1-\sqrt{x}}>\frac{1-\sqrt{x}}{1+\sqrt{x}}\,.$

Functions, sequences and greatness preserving estimates

7. What is the greatest and the least value of the following function:

$$f(x)=\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}} \quad (x\in[2;17]).$$

8. Dividing by the dominant term give the following expressions as a function of $\dfrac{1}{n}$, that is in a form of $f\left(\dfrac{1}{n}\right)$:

(a) $\dfrac{5n^3 - 3n^2 + 2n + 7}{8n^3 + 7n - 3}$;

(b) $\dfrac{\sqrt{n+1} + 3 \cdot \sqrt{n}}{\sqrt{2n-1}}$;

(c) $\dfrac{\sqrt[3]{(n+1)^2 + 2}}{\sqrt[3]{n^2} - 1}$.

9. Rationalize the following expressions and then write them in a form of a function depending on $\dfrac{1}{n}$:

(a) $\sqrt{n^2 + n + 1} - \sqrt{n^2 + 1}$;

(b) $\dfrac{1}{\sqrt{n^2 + n} - \sqrt{n^2 - n - 1}}$;

(c) $\sqrt[3]{n} \cdot (\sqrt[3]{n^2 + n + 2} - \sqrt[3]{n^2 + 1})$.

10. Consider the following sequences (x_n). Give the following new sequence (quotient-sequence) $\left(\left|\dfrac{x_{n+1}}{x_n}\right|\right)$:

(a) $x_n = \dfrac{n!}{2^{n+1}} \qquad (n \in \mathbb{N}^+)$;

(b) $x_n = \dfrac{(2n)!}{(n!)^2} \qquad (n \in \mathbb{N}^+)$;

(c) $x_n = \dfrac{3^n \cdot n^2}{(n+1)!} \qquad (n \in \mathbb{N}^+)$;

(d) $x_n = \dfrac{n^n \cdot (-1)^{n+1}}{(2n+1)!} \qquad (n \in \mathbb{N}^+)$;

(e) $x_n = \dfrac{\sqrt{3n+2}}{(1+\sqrt{1}) \cdot (1+\sqrt{2}) \cdot \ldots \cdot (1+\sqrt{n})} \qquad (n \in \mathbb{N}^+)$.

11. Consider the following sequences (x_n). Give the most simple form of the sequence $(\sqrt[n]{|x_n|})$:

(a) $x_n = \dfrac{2^{n+1}}{(n^2+1)^{2n}} \qquad (n \in \mathbb{N}^+)$;

(b) $x_n = \dfrac{(-1)^n}{2^{1-n} + 2^n} \qquad (n \in \mathbb{N}^+)$;

(c) $x_n = \left(1 + \frac{1}{n}\right)^{n^2+n+1} \qquad (n \in \mathbb{N}^+)$.

3.2.2. Homework and more exercises to practice

Algebraic computations and simplifications

1. Simplify the following algebraic expressions:

(a) $\dfrac{x^2+2x-3}{5x^2+16x+3}$;

(b) $\dfrac{x^4+8x^2+15}{x^4+6x^2+9}$;

(c) $\dfrac{27x^3-1}{6x^2+x-1}$;

(d) $\dfrac{x^4+x^2+1}{x^3+1}$.

2. Rationalize the radicals and then simplify the following expressions if it is possible:

(a) $\dfrac{x^2-1}{\sqrt{x^3+5}-2}$;

(b) $\dfrac{x^2-9}{\sqrt{\sqrt{x}-\sqrt{3}+4}-2}$;

(c) $\dfrac{x^2-26x-27}{\sqrt[3]{x}-3}$.

Equations and inequalities containing absolute values

3. Solve the following equations and inequalities on the set of the real numbers:

(a) $\left|\dfrac{3|x|-2}{|x|-1}\right| = 2$;

(b) $||x+1|-2| = ||x-2|+1|$;

(c) $|x+3|+|x-1| = 3x-5$;

(d) $|x+1|-|x|+3|x-1|-2|x-2| = x+2$;

(e) $|x+3|+\sqrt{x^2-2x+1} = 8$;

(f) $\left|\dfrac{x}{1+x}-\dfrac{2}{3}\right| \le \dfrac{|x-2|}{1+|x-2|}$;

(g) $x^2-6|x|-7<0$;

(h) $\left|\dfrac{x+1}{x-1}\right|+\left|\dfrac{x-1}{x+1}\right| \le 2$;

(i) $|\,|x+1|-|x-1|\,|<1$;

(j) $|x|>|x-1|$;

(k) $|x+2|-|x|\ge 1$.

4. Prove the following statements:

(a) $0<a+b-ab<1 \quad (a,b\in(0,1))$;

(b) $a^2+b^2\ge 2|ab| \quad (a,b\in\mathbb{R})$;

(c) $2x^4-2x^3-x^2+1\ge 0 \quad (x\in\mathbb{R})$;

(d) $ab-5a^2-3b^2\le 0 \quad (a,b\in\mathbb{R})$;

(e) $|a+b|<|1+ab| \quad (a,b\in\mathbb{R},\ |a|,|b|<1)$;

(f) $|a+b|+|a-b|\ge |a|+|b| \quad (a,b\in\mathbb{R})$;

(g) $|a|+|b|+|c|+|a+b+c|\ge |a+b|+|b+c|+|c+a| \quad (a,b,c\in\mathbb{R})$;

(h) $\left(\dfrac{a+2b}{a+b}\right)^2-2<2-\left(\dfrac{a}{b}\right)^2 \quad (a,b\in(0,+\infty),\ a^2<2b^2)$.

5. If the real number x satisfies the condition $|x+5|<3$ find lower and upper bounds for $1/|x-2|$.

6. Suppose that the real number x satisfies the condition $|x+1|<1/2$. Give an upper estimate for the expression $|1/x^2-1|$.

Equations and inequalities containing radicals, roots

7. Solve the following equations and inequalities:

(a) $\sqrt{5+\sqrt{x+1}}+\sqrt{3-\sqrt{x+1}}=\sqrt{7}+1$;

(b) $\sqrt{3+\sqrt{5-x}}=\sqrt{x}$;

(c) $\sqrt{2x+3}+\sqrt{3x+3}=1$;

(d) $\sqrt{4x^2+4x+1}-\sqrt{4x^2-12x+9}=4$;

(e) $\sqrt{\dfrac{x-3}{2}}+\sqrt{2x}=\sqrt{x+3}$;

(f) $\sqrt{x+2+2\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=2$;

(g) $\sqrt{3x^2-|x|-1}=3-2x$;

(h) $\sqrt{3x+10} \le x+4$;

(i) $\sqrt{3x+7} < x-1$;

(j) $\sqrt{5x+16} > x+2$;

(k) $\sqrt{x-5} - \sqrt{x} \le 5$;

(l) $\sqrt{x-5} - \sqrt{x} \le 5$.

8. Are there any rational numbers x so that:

(a) $\dfrac{\sqrt{x-1}}{\sqrt{x-10}} = \dfrac{\sqrt{3x+22}}{\sqrt{3x-14}}$;

(b) $\sqrt{4-2\sqrt{x^2-1}} = 2x$?

9. Prove that the following inequality holds for all $1 \le n \in \mathbb{N}$:

$$\frac{1}{\sqrt{n}} < \sqrt{n+1} - \sqrt{n-1}.$$

10. Prove that for all $1 \le n \in \mathbb{N}$ we have:

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}.$$

Functions, sequences and greatness preserving estimates

11. Dividing by the dominant term write the following expressions in a form of a function depending on $\frac{1}{n}$ so in a form of $f\left(\frac{1}{n}\right)$:

(a) $\dfrac{3n^4+5n^3-7n+4}{5n^4-10n^2+2}$;

(b) $\dfrac{\sqrt{2n^2+5n}}{\sqrt{n^2+6}+3n+1}$;

(c) $\dfrac{\sqrt[3]{n^2+n+1}+\sqrt[3]{n^2+6}}{\sqrt[3]{n^2+1}+\sqrt[3]{n^2-1}}$.

12. Rationalize the following expressions and then write them in a form of a function of $\frac{1}{n}$:

(a) $\sqrt{n^4+n^2+5} - \sqrt{n^4-2n^2-7}$;

(b) $\dfrac{\sqrt{n}}{\sqrt{n^3+n^2+2}-\sqrt{n^3-n^2+3}}$;

(c) $\sqrt[3]{n^3+n^2+3}-n\,.$

13. Consider the following sequences (x_n) . Give the quotient-sequence $\left(\left|\frac{x_{n+1}}{x_n}\right|\right)$:

(a) $x_n = \dfrac{5^{n+2}}{(n+1)!} \qquad (n \in \mathbb{N}^+)$;

(b) $x_n = \dfrac{(3n)!}{(n!)^3} \qquad (n \in \mathbb{N}^+)$;

(c) $x_n = \dfrac{(n+2)!}{4^n \cdot (n^2+3)} \qquad (n \in \mathbb{N}^+)$;

(d) $x_n = \dfrac{(-n-2)^n}{(2n+2)!} \qquad (n \in \mathbb{N}^+)$.

14. Consider the following sequences (x_n). Evaluate the new sequence $(\sqrt[n]{|x_n|})$:

(a) $x_n = \dfrac{3^{2n-1}}{(n^3+1)^{5n}} \qquad (n \in \mathbb{N}^+)$;

(b) $x_n = \dfrac{(-1)^{n+1}}{3^{2-n}+3^n} \qquad (n \in \mathbb{N}^+)$;

(c) $x_n = \left(1-\dfrac{1}{n}\right)^{3n^2-n} \qquad (n \in \mathbb{N}^+)$.

4. Exponential and logarithmic expressions, equations and inequalities

4.1. Theoretic review

It is highly recommended to look through the following topics:

1. Exponential computations, expressions.
2. Exponential functions and their properties, graphs.
3. Exponential equations and inequalities.
4. The definition of the logarithm and its properties.
5. Logarithmic functions and their properties, graphs.
6. Logarithmic equations and inequalities.

4.1.1. Checking questions to the theory and its use

1. What is the definition of $\log_a b$ and what conditions should satisfy a and b ?
2. For what real number x is true that: $2^x = 3$?
3. What real numbers x satisfy that : $\dfrac{1}{2^{\sqrt{3x}}} = 4^{-3/2}$?
4. Give the formula to evaluate the logarithm of a product and give the necessary conditions as well.
5. Give the formula related to the logarithm of a quotient and the needed conditions as well.
6. Write the number $\log_5 2$ in a form of logarithms with base 3.
7. Simplify the following expression:

$$13^{\frac{\lg(\ln 13)}{\lg 13}}.$$

8. Give the most simple form of the following expression:

$$\sqrt[4]{x^{4+\log_x 36}} \quad (1 \neq x \in (0; +\infty)).$$

9. Solve the following equation:

$$\ln^2 x - \ln x^3 + \ln e^2 = 0.$$

10. Solve the following equation:

$$x + \log_2(9 - 2^x) = 3.$$

11. Solve the following inequality:

$$|\lg(x-1) - 10| < 1.$$

12. Draw the graph of the following function:

$$f(x) := e^{1-x} \quad (x \in \mathbb{R}).$$

13. Draw the graph of the function:

$$f(x) := 3^{|x|+1} \quad (x \in \mathbb{R}).$$

14. Draw the graph of the function:

$$f(x) := \log_2 x^2 \quad (x \in \mathbb{R} \setminus \{0\}).$$

15. Find the minimal and maximal values of f and their place, if:

$$f(x) := \sqrt{\ln x + 1} - \sqrt{\ln x - 1} \quad (x \in [e; e^2]).$$

4.2. Exercises

4.2.1. Exercises for class work

Exponential equations and inequalities

1. Solve the following equations and inequalities:

$$2^x = 128; \qquad 2^x \geq 128; \qquad 2^x < 128; \qquad \left(\frac{1}{27}\right)^x = 81; \qquad \left(\frac{3}{5}\right)^x > \frac{25}{9}.$$

2. Solve the following equation on the set of the real numbers:

$$2^{x+3} + 4^{1-x/2} = 33.$$

3. Solve the following equations and inequalities on the set of the real numbers:

(a) $9 \cdot 3^{x-2} + 6 \cdot 3^{x-1} + 5 \cdot 3^x = 2 \cdot 3^{x+1} + 18$;

(b) $16 \cdot 3^x = 9 \cdot 2^{2x}$;

(c) $3^{x+2} \cdot 2^x - 2 \cdot 36^x + 18 = 0$;

(d) $3 \cdot 4^x + \frac{1}{3} \cdot 9^{x+2} = 6 \cdot 4^{x+1} - \frac{1}{2} \cdot 9^{x+1}$;

(e) $\sqrt{(17 - 12\sqrt{2})^x} + \sqrt{(17 + 12\sqrt{2})^x} = \frac{10}{3}$;

(f) $4^{x+1} - 9 \cdot 2^x + 2 > 0$.

4. Using the substitution $x = \dfrac{e^t + e^{-t}}{2} (t \in \mathbb{R})$ solve the equation:

$$4x^3 - 3x - a = 0 \quad (a > 1).$$

5. Solve the inequality:

$$e^x + e^{-x} > 3.$$

6. Find the least and the greatest values of the function f.

$$f(x) := \frac{1}{\sqrt{2^x + 2} - \sqrt{2^x - 2}} \quad (x \in [1; 2]).$$

Where does f take these values?

Computations with logarithms, logarithmic equations and inequalities

7. Determine the exact value of:

$$5^2 \cdot 5^{\log_{25} 36 - 1} + 5^{1 + \log_{125} 8}.$$

8. Determine the exact value of:

$$3^{2 + \log_9 25} + 25^{1 - \log_5 2} + 10^{-\lg 4}.$$

9. Give the most simple form of the following expression:

$$\log_x \sqrt[3]{x^2 \cdot \sqrt{\frac{1}{y} \cdot \frac{1}{\sqrt{x^{-1} \cdot y^{-1}}}}} \quad (1 \neq x \in (0; +\infty); y > 0).$$

10. Determine the exact value of:

$$\frac{1}{2} \cdot \lg 52 + 3 \cdot \lg 2 + \lg 125 + \lg \sqrt{325} - \lg 13.$$

11. Find x expressed with a, b, c, if

$$\log_a x = 3 \cdot (\log_a b - \log_{a^2} c) \quad (1 \neq a > 0; b, c > 0).$$

12. Knowing that $\log_{16}(54) = a$ express with a the value of $\log_{12}(18)$.

13. Knowing that $20x^2 - y^2 + 8xy = 0$ is valid evaluate the exact value of:

$$\lg x - \lg y.$$

14. Solve the following equations and inequalities:

$$\log_5 x = -1; \qquad \log_5 x \leq -1; \qquad \log_5 x \geq -1;$$

$$\log_{\frac{1}{3}} x < -2; \qquad \log_{\frac{1}{3}} x > -2; \qquad \log_{\frac{1}{3}} x = -2.$$

15. Solve the following equations and inequalities on the set of the real numbers:

(a) $\log_3 (\log_2 (\lg(2x))) = 0;$

(b) $\log_{25} \left[\frac{1}{5} \cdot \log_3 \left(2 - \log_{\frac{1}{2}} x\right)\right] = -\frac{1}{2};$

(c) $\log_3(x+1) - \log_3(x+10) = 2\log_3 4,5 - 4;$

(d) $\log_2(x-2) + \log_2(x+3) = 1 + 2\log_4 3;$

(e) $\log_{32}(2x) - \log_8(4x) + \log_2(x) = 3;$

(f) $\log_x(8) - \log_{4x}(8) = \log_{2x}(16);$

(g) $x^{(2\cdot\lg^2 x - 1,5\cdot\lg x)} = \sqrt{10}\,;$

(h) $\log_3 \frac{3x-1}{x+2} > 1;$

(i) $\log_3 \frac{3x-1}{x+2} < 1;$

(j) $\log_{\frac{1}{2}} \left(\frac{3-x}{3x-1}\right) \geq 0;$

(k) $\log_{\frac{1}{2}} \left(\frac{3-x}{3x-1}\right) \leq 0;$

(l) $\log_x(\lg(x)) > 0;$

(m) $\log_{1/x} \frac{2x-1}{x-1} \leq -1.$

16. Find the minimal and maximal values of the following function and give their place too:

$$f : [1; 64] \to \mathbb{R} \quad f(x) = (\log_2 x)^4 + 12 \cdot (\log_2 x)^2 \cdot \log_2 \frac{8}{x}.$$

17. For what real numbers x can we define the following expression:

$$E(x) = \frac{\sqrt{2 - \sqrt{1-x}}}{\ln(x^2 - 1)}\ ?$$

18. Draw the graph of the following function:

$$f(x) := \ln \frac{1}{x} \quad (x \in (0; +\infty)).$$

19. Prove that for all positive numbers a, b we have:

$$\ln \frac{a+b}{2} \geq \frac{\ln a + \ln b}{2}.$$

What is the geometric meaning of this inequality?

20. Prove that for all real numbers a, b we have:

$$\frac{e^a + e^b}{2} \geq e^{(a+b)/2}.$$

What is the geometric meaning of this inequality?

4.2.2. Homework and exercises to practice

Exponential equations and inequalities

1. Find all the real numbers $x \in \mathbb{R}$ for which:

(a) $8^{x-1} - 2^{3x-2} + 8 = 0$;

(b) $2^{3x+1} + 3^{2x+2} = 11$;

(c) $2^{x+1} \cdot 5^{x+1} = 0.01^{-x}$;

(d) $2^x - 0.5^x = 3.75$;

(e) $3^x + 3^{-x} = p \quad (p \in \mathbb{R}$ is a parameter);

(f) $(x-1)^x = \sqrt[3]{x-1}$;

(g) $5^{3x-4} < \dfrac{1}{25}$;

(h) $\dfrac{3}{7}^{4-3x} \geq \dfrac{49}{9}$;

(i) $2^x + 2^{1-x} < 3$.

Computations with logarithms, logarithmic equations and inequalities:

2. Knowing that $\log_{12}(27) = a$ express with a the value of: $\log_6(16)$.

3. Solve the equation:

$$\ln(10x) = \lg(ex).$$

4. Solve the equation:

$$x^{\ln x} = e.$$

5. Solve the equation:

$$(2^x + 2)^{1/x} = 4.$$

6. Solve the following equations and inequalities on the set of the real numbers:

(a) $\log_{4-x}(x^2+16) + \log_{4-x}(2x+1) = \log_{4-x}(2x) + \log_{4-x}(x^2+21)$;

(b) $\log_{x+1}(2x^2+8x+6) = 2$;

(c) $\log_3(x^3-1) - \log_3(x^2+x+1) = 2$;

(d) $\lg(x^4) + \lg(x^2) = 6$;

(e) $\log_{x-1}(x^2-2x+1) = 2$;

(f) $\lg(x+24) = 2 - 2\lg(\sqrt{x+3})$;

(g) $4\log_a(x) - 4\log_{a^2}(x) + 4\log_{a^4}(x) = 3 \quad (a \in \mathbb{R})$;

(h) $\lg(x) + \lg(x-3) = 1$;

(i) $2 \cdot \lg(2) + \lg(2x+1) - \lg(-12x) = \lg(1-2x)$;

(j) $\dfrac{\log_3(2x)}{\log_3(4x-15)} = 2$;

(k) $\log_x(x^3+3x^2-27) = 3$;

(l) $\log_2(x) + 2 \cdot \log_4(x) = 3 \cdot \log_8(x) + 1$;

(m) $(\log_2(x)) \cdot (\log_4(2x)) = 2 \cdot \log_4(2)$;

(n) $\log_2(17-2^x) + \log_2(2^x+15) = 8$;

(o) $x^{\log_{x^2}(x^2-1)} = 5$;

(p) $\sqrt{x^{\lg\sqrt{x}}} = 10$;

(q) $x^{\lg(x)} = 0.1 \cdot x^2$;

(r) $\log_3\left(\log_3^2(x) - 3 \cdot \log_2(x) + 5\right) = 2$;

(s) $\log_{\frac{1}{5}}(x^2+x-30) < 0$;

(t) $\log_{1-x}\dfrac{2x+3}{4(2x+1)} \geq 1$;

(u) $\log_3 x \geq \log_x 3$.

7. Solve the following inequality:

$$\log_{1/2}(x^2-1) + \log_2(x-1) < \log_4(x+1).$$

8. Solve the following inequality:

$$\ln(x-1) - \ln(x+1) \geq -\ln x.$$

9. Solve the following inequality:

$$(\ln x)^x > e^x.$$

10. Prove that:

$$\frac{1}{\log_2(\pi)} + \frac{1}{\log_\pi(2)} > 2.$$

11. Prove that:

$$\log_a(a^2+1) + \log_{a^2+1}(a^2) > 3 \quad (a \in (1, +\infty)) .$$

12. Prove that, if $a, b \in (0; 1)$ then:

$$\log_a \frac{2ab}{a+b} + \log_b \frac{2ab}{a+b} \geq 2.$$

13. Prove that:

$$\log_2 3 + \log_3 4 + \log_4 5 + \log_5 6 > 5.$$

14. For what real numbers x can we define the following expression:

$$E(x) = \frac{\sqrt{2^x - 1}}{\log_x(|x| - 3)} \ ?$$

15. Consider the parameter $a \in (0; 1)$ and the function $f(x) := a^x + (1-a)^x \ \ (x \in \mathbb{R})$. Prove that:

(a) If $x > 1 \implies f(x) < 1$;

(b) If $x < 1 \implies f(x) > 1$.

16. Prove that for all $a \neq b \in (0; +\infty)$ we have:

$$a^a \cdot b^b > a^b \cdot b^a.$$

17. Prove that for all $a \neq b \in (0; +\infty)$ and $\alpha \in (0; 1)$ we have:

$$a^\alpha \cdot b^{1-\alpha} < a + b.$$

18. Prove that:

$$\exists!\, x \in [1; +\infty) : 1 + 2 \cdot \ln x = e^{1-x}.$$

19. Find the minimal and maximal value of the following function f and give their place too:

$$f(x) := \log_2^2 x + \log_x^2 2 \quad (x \in (0; +\infty) \setminus \{1\}).$$

20. For what real numbers x, y holds the following inequality:

$$\sqrt{\log_x(\pi - \sqrt{y})} + 2\cos(3\pi \cos \sqrt{y}) + \sqrt{\log_{\pi - \sqrt{y}} x} \leq 0?$$

5. Trigonometric computations, equations and inequalities

5.1. Theoretic review

It is recommended to review the following topics:

1. Measuring angles (degree, radian).
2. Definition of trigonometric functions (sin, cos, tg, ctg).
3. Important trigonometric values.
4. The graphs and properties of the following trigonometric functions:

$$\sin, \cos, \mathrm{tg}, \mathrm{ctg}.$$

5. Some basic trigonometric identities:

   (a) $\sin^2 \alpha + \cos^2 \alpha = 1$;

   (b) $\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$;

   (c) $\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$;

   (d) $\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$;

   (e) $\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$;

   (f) $\sin 2\alpha = 2 \sin \alpha \cdot \cos \alpha$;

   (g) $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$;

   (h) $\sin^2 \alpha = \dfrac{1 - \cos 2\alpha}{2}$, $\cos^2 \alpha = \dfrac{1 + \cos 2\alpha}{2}$ (*Linearization formulas*).

   Here $\alpha, \beta \in \mathbb{R}$ are real numbers (angles given in radian).

5.1.1. Checking questions to the theory and its use

1. Give the angle of $120°$ in radian.
2. Give the exact value of:

$$\sin \pi/3; \ \cos \pi; \ \sin \pi/4; \ \cos \pi/2; \ \mathrm{tg}\, \pi/4; \ \mathrm{ctg}\, \pi/6; \ \mathrm{tg}\, \pi/3.$$

3. Evaluate the exact value of:

$$(\sin \pi/7 + \cos \pi/7)^2 - \sin 2\pi/7.$$

4. Evaluate $\sin^3 \pi/3 - \cos^3 \pi/3$.

5. For what real numbers $a \in \mathbb{R}$ holds the following equality for all the given real numbers x:
$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{a}{\sin^2(2x)} \quad (\forall\, x \in \mathbb{R} \setminus \{k\pi/2 \mid k \in \mathbb{Z}\})?$$

6. Evaluate $\sin(x - y)$ using the additional theorem.

7. Evaluate $\cos(x + y)$ using the proper formula.

8. Determine the exact value of the following expression:
$$\sin \pi/7 \cdot \cos \pi/42 + \sin \pi/42 \cdot \cos \pi/7.$$

9. Determine the exact value of $\sin \pi/8$.

10. Solve the following equation on the set of the real numbers:
$$\cos^2 x = 1 + \sin^2 x.$$

11. Draw the graph of the function $f(x) := \sin^4 x - \cos^4 x \quad (x \in [0; \pi])$.

12. Give the most simple form of the following expression:
$$E(x) := (\sin x + \cos x)^4 - (\sin x - \cos x)^4 \quad (x \in \mathbb{R}).$$
For what real numbers x do we have:
$$E(x) = -2\,?$$

13. Solve the following inequality on the interval $[\pi/2; \pi]$:
$$\sin 2x > \cos x.$$

14. Find the *largest* set D for which f is a function:
$$f(x) := \sqrt{\sin x} + \frac{1}{\sqrt{\sin x}} \quad (x \in D).$$
What is the minimal value of f and where does f take this value.

15. Determine the exact value of:
$$\frac{\sin^2 \dfrac{4037\pi}{4}}{1 - \cos^3 7\pi}.$$

16. Draw the graph of the function $f(x) := \sin x \quad (x \in [0; 3\pi])$.

17. Draw the graph of the following function $g(x) := \cos x \quad (x \in [-\pi; 3\pi])$.

18. Define and draw the graph of the function tg.

19. Write down the *linearization* formulas.

20. Determine the exact value of:
$$\cos 9\pi/20 \cdot \cos \pi/5 + \sin 9\pi/20 \cdot \sin \pi/5.$$

5.2. Exercises

5.2.1. Exercises for class work

Trigonometric identities, equations

1. Evaluate the exact value of $\operatorname{tg} \pi/12$.

2. Give a linearization formula for $\cos^3 \alpha$, if $\alpha \in \mathbb{R}$.

3. Solve the following equations on the set of the real numbers:

$$\sin x = -\frac{1}{2}; \qquad \sin\left(x + \frac{\pi}{7}\right) = \frac{\sqrt{3}}{2}; \qquad \cos\left(3x - \frac{\pi}{4}\right) = \frac{1}{2}.$$

$$\operatorname{tg} x = \sqrt{3}; \qquad \operatorname{tg}\left(2x - \frac{2\pi}{3}\right) = -1; \qquad \operatorname{ctg}^2\left(2x - \frac{\pi}{5}\right) = \frac{1}{3}.$$

4. For what real numbers $x \in \mathbb{R}$ do we have:

(a) $\sin 4x = \sin x$;

(b) $\cos 10x = \cos 2x$;

(c) $\cos 4x = \sin 3x$;

(d) $\cos 2x - 3\cos x + 2 = 0$;

(e) $\operatorname{ctg} x - \operatorname{tg} x = 2\sqrt{3}$;

(f) $\dfrac{\cos x}{\operatorname{tg} x} = \dfrac{3}{2}$;

(g) $\dfrac{4}{\cos^2 x} - 5\operatorname{tg}^2 x = 1$;

(h) $\sqrt{3} \cdot \sin x + \cos x = \sqrt{3}$;

(i) $\sqrt{2} \cdot \sin x \cos \dfrac{x}{2} = \sqrt{1 + \cos x}$;

(j) $2\sin^2 x + \sin x + \dfrac{1}{2\sin^2 x} + \dfrac{1}{2\sin x} = 1$;

(k) $\sin^2 x + \dfrac{1}{4}\sin^2 3x = \sin x \cdot \sin^2 3x$;

(l) $9^{\sin^2 x} + 9^{\cos^2 x} = 6$?

(m) $\cos 2x = \cos x - \sin x$?

Inequalities

5. Solve the following inequalities on the set of the real numbers:

$$\sin x < -\frac{1}{2}; \qquad \sin x > -\frac{1}{2}; \qquad \cos x \le -\frac{1}{2}; \qquad \cos x \ge -\frac{1}{2}.$$

6. For what numbers $x \in \mathbb{R}$ is true that:

$$-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}\,?$$

7. Find all the real numbers $x \in \mathbb{R}$ for which:

(a) $2\sin^2 x - \sin x - 1 > 0\,;$

(b) $2\cos^2 x + \sin x - 1 < 0\,;$

(c) $\dfrac{2\sin x + 1}{2\cos x} \leq 0\,;$

(d) $\dfrac{\mathrm{tg}^2 x - \sin^2 x}{\mathrm{ctg}^2 x - \cos^2 x} > 1\,;$

(e) $\left|\dfrac{\sin x - \cos x}{\sin x + \cos x}\right| \leq 1\,.$

Some other types

8. a) Simplify the following expression, where x denotes real number for which the following expression is well defined:

$$E(x) = \frac{\sin\left(\frac{5\pi}{2} + x\right) + \cos 3x + \sin\left(\frac{\pi}{2} - 5x\right)}{\sin 3x - \cos\left(\frac{\pi}{2} + x\right) + \sin 5x}.$$

b) Solve the following equation:

$$E(x) + \frac{1}{E(x)} = \frac{4}{\sqrt{3}}.$$

9. Prove that the following function is constant on the given interval:

$$f(x) = \sqrt{\cos^2 x + \sqrt{\cos 2x}} + \sqrt{\cos^2 x - \sqrt{\cos 2x}} \quad (x \in [-\pi/4; \pi/4]).$$

10. Find the largest set D so that the following expression is well defined for all numbers $x \in D$:

$$f(x) = \frac{\sin^2\left(\frac{3\pi}{8} - x\right) - \sin^2\left(\frac{\pi}{8} - x\right)}{\sin\left(\frac{\pi}{4} - x\right)} \quad (x \in D).$$

Solve the following equation:

$$f(x) + (2 + \sqrt{3})f(-x) = 0.$$

11. Using the properties of the cos function find the range R_f (set of the values of f) of the following function:

$$f(x) := \cos\frac{1}{x} \quad \left(x \in \left[\frac{3}{2\pi}; \frac{2}{\pi}\right)\right)?$$

12. Give the largest set D so that the following function is well defined:

$$f(x) := (\sqrt{\mathrm{tg}} - \sqrt{\mathrm{ctg}\, x})^2 \quad (x \in D).$$

What are the minimal and maximal values of f on the interval $[\pi/8; 5\pi/12]$? Where does f take them?

13. Evaluate the minimal and maximal values of the function

$$f(x) := \sin^2\frac{x}{2} \cdot \cos^2\frac{x}{2} \quad (x \in [0; \pi/4]).$$

Where does f take them?

14. Prove the following identity for all $1 \leq n \in \mathbb{N}$ and real numbers $x \neq \dfrac{\lambda\pi}{2^k}$ $(k = 0, 1, 2, 3, \ldots, n; \lambda \in \mathbb{Z})$:

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \frac{1}{\sin 8x} + \cdots + \frac{1}{\sin 2^n x} = \mathrm{ctg}\, x - \mathrm{ctg}\, 2^n x.$$

15. Give an explicit formula to evaluate x_n , where $1 \leq n \in \mathbb{N}$ and in this expression we have n square roots:

$$x_n = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}}}} \quad (1 \leq n \in \mathbb{N}).$$

5.2.2. Homework and exercises to practice

Trigonometric identities, equations

1. Express $\mathrm{tg}\,(x + y)$ using $\mathrm{tg}\, x$ and $\mathrm{tg}\, y$.

2. What is the exact value of $\mathrm{tg}\, \pi/8$.

3. Determine the exact value of:

$$\mathrm{tg}\, \pi/16 + \mathrm{ctg}\, \pi/16.$$

4. Give a linearization formula for $\sin^3\alpha$, where $\alpha \in \mathbb{R}$.

5. Prove the following identities for all the real numbers x for which both sides of the following identities are well defined:

(a) $\sin^4 x + \cos^4 x = 1 - \dfrac{\sin^2 2x}{2}$;

(b) $\sin^4 x + \cos^4 x = \dfrac{3}{4} + \dfrac{1}{4}\cos 4x$;

(c) $\dfrac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$;

(d) $\operatorname{tg}^2 x - \sin^2 x = \operatorname{tg}^2 x \cdot \sin^2 x$;

(e) $\sin 2x = \dfrac{2\operatorname{ctg} x}{1 + \operatorname{ctg}^2 x}$.

6. Find those real numbers $x \in \mathbb{R}$ for which we have:

(a) $4\cos^3 x + 3\cos(\pi - x) = 0$;

(b) $\operatorname{tg} x + \operatorname{ctg} x = \dfrac{2}{\sin 2x}$;

(c) $\sin x = \sqrt{3} \cdot \cos x$;

(d) $\sin^2 x - 2\cos x \cdot \sin x - 3\cos^2 x = 0$;

(e) $\sin x \cdot \operatorname{tg} x = \dfrac{1}{2\sqrt{3}}$;

(f) $2\cos 2x + 4\sin x + 1 = 0$;

(g) $\sin x + \sqrt{3} \cdot \cos x = 2$.

7. Prove that for all $x \in \mathbb{R} \setminus \{m\pi \mid m \in \mathbb{Z}\}$ and $n \in \mathbb{N}$ we have:

$$\prod_{k=0}^{n} \cos(2^k \cdot x) = \frac{\sin(2^{n+1}x)}{2^{n+1}\sin x} .$$

8. Solve the following equation on the set of the real numbers:

$$3(\log_2 \sin x)^2 + \log_2(1 - \cos 2x) = 2.$$

9. Solve the following equation:

$$1 + 2^{\operatorname{tg} x} = 3 \cdot 4^{f(x)},$$

where $f(x) = \dfrac{\sin(\pi/4 - x)}{\sqrt{2} \cdot \cos x}$.

10. Solve the following equation:

$$2^{\cos^2 x} = \sin x.$$

Inequalities

11. Find all the real numbers $x \in \mathbb{R}$ so that:

$$\cos x < \cos^4 x.$$

12. Find all the real numbers $x \in \mathbb{R}$ so that:

(a) $|\sin x - \cos x| \le \sqrt{2}$;

(b) $\sin^4 x + \cos^4 x \ge \dfrac{1}{2}$;

(c) $\dfrac{1}{4} \le \sin^6 x + \cos^6 x \le 1$;

(d) $\sin x \cos 6x > \cos x \sin 6x$;

(e) $\sin^2 x - \dfrac{\sqrt{3}+\sqrt{2}}{2} \cdot |\sin x| + \dfrac{\sqrt{6}}{4} < 0.$

Other types

13. a) Simplify the following fraction:

$$E(x) = \frac{\sin 6x - \cos\left(\frac{\pi}{2} + 4x\right) + 2\sin x \cos x}{\sin\left(\frac{9\pi}{2} + 2x\right) + \cos 6x + \sin\left(\frac{\pi}{2} + 4x\right)}.$$

b) Solve the equation:

$$E(x) - \frac{1}{E(x)} = -2.$$

14. Give the largest set D so that the following function is well defined:

$$f(x) := \sqrt{\sin x} + \frac{1}{2\sqrt{\cos x}} \quad (x \in D).$$

Find the minimal and maximal values of f on the interval: $[0; \pi/4]$. Where does f take them?

15. What are the minimal and maximal values of f and where does it take them:

$$f(x) := \sin\frac{\pi}{x} \cdot \cos\frac{x}{\pi} + \sin\frac{x}{\pi} \cdot \cos\frac{\pi}{x} \quad (x \in [\pi; 2\pi])?$$

16. Draw the graph of $f(x) := \sin^4 x + \cos^4 x \quad (x \in [0; \pi/2])$.

17. Find the minimal and maximal value of f:

$$f(x) = \mathrm{tg}\frac{\pi\cos^2 x}{4} + \mathrm{tg}\frac{\pi\sin^2 x}{4} \quad (x \in \mathbb{R}).$$

18. The angles α, β, γ of a triangle have the property, that $\mathrm{ctg}\frac{\alpha}{2}, \mathrm{ctg}\frac{\beta}{2}, \mathrm{ctg}\frac{\gamma}{2}$ are consequtive natural numbers. Find the largest angle of the triangle.

6. Order preserving estimates

6.1. Theory

Polynomials

Let $n \in \mathbb{N}$ be a natural number and a real polinomal of degree n is defined as follows:

$$P(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0 \quad (x \in \mathbb{R}),$$

where a_n, a_{n-1}, ..., a_1, a_0 are real numbers (the *coefficients* of P) and $a_n \neq 0$. Here a_n is called the main coefficient of P and x denotes the *variable* of P which can be any real number. When $n = 0$ we have a constant polynomial, with the value a_0 for all real numbers x. For the constant $P = 0$ polynomial: $P(x) = 0 \quad (x \in \mathbb{R})$ we do not define its degree and its main coefficient.

The order preserving estimates of polynomials

Consider a real polynomial of degree n with a positive main coefficient ($a_n > 0$).
We would like to examine the behaviour of the values $P(x)$ when x takes *great enough* positive values. One can make the intuition that when $0 < x$ is great, then the value $P(x)$ is determined by its dominant term: $a_n \cdot x^n$, and the "rest" of $P(x)$ is negligible. More precisely this means that if:

$$P(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0 \qquad (a_n > 0, x \in \mathbb{R}),$$

we can find some fixed positive numbers $R > 0$, $m > 0$, $M > 0$ so that for all $x \geq R$ we have:

$$m \cdot x^n \leq P(x) \leq M \cdot x^n \quad (x \in [R; +\infty)).$$

So for all great enough positive numbers x the values $P(x)$ are between the values of some constants times x^n.

The polynomial

$$L(x) := m \cdot x^n \quad (x \in \mathbb{R})$$

(by giving the number $R > 0$ as well) is called the *order preserving lower estimate* (OPL-estimate) of P and the polynomial

$$U(x) := M \cdot x^n \quad (x \in \mathbb{R})$$

(by giving the number $R > 0$ as well) is called the *order preserving upper estimate* (OPU-estimate) of P. These two estimates toegether are called the *order preserving estimates* or shortly *OP-estimates* of P.

The method of finding the constants $R > 0$, $m > 0$, $M > 0$ will be presented at class.

The estimates of rational expressions

Consider the polynomials

$$P_1(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0 \qquad (a_n \neq 0, x \in \mathbb{R})$$

and

$$P_2(x) = b_k \cdot x^k + b_{k-1} \cdot x^{k-1} + \ldots + b_1 \cdot x + b_0 \qquad (b_k \neq 0, x \in \mathbb{R})$$

and define their ratio by:

$$R(x) = \frac{P_1(x)}{P_2(x)} \quad (x \in \mathbb{R} \setminus \{x \in \mathbb{R} \mid P_2(x) = 0\}).$$

We call the ratio of two polynomials a *rational function* or *rational expression* or a *rational fraction*.We can also give order preserving OP-estimates for such type of rational expressions R. Suppose that n and k are the degrees of the given polynomials, $a_n, b_k > 0$ and we already have the OP-estimates for the polynomials P_1 and P_2 , so there exist positive real numbers $R_1, m_1, M_1 > 0$ and $R_2, m_2, M_2 > 0$ so that:

$$m_1 \cdot x^n \leq P_1(x) \leq M_1 \cdot x^n \qquad (x \geq R_1),$$

and

$$m_2 \cdot x^k \leq P_2(x) \leq M_2 \cdot x^k \qquad (x \geq R_2).$$

If $x \geq R := \max\{R_1, R_2\}$ then we have:

$$R(x) = \frac{P_1(x)}{P_2(x)} \leq \frac{M_1 \cdot x^n}{m_2 \cdot x^k} = \frac{M_1}{m_2} \cdot x^{n-k},$$

and

$$R(x) = \frac{P_1(x)}{P_2(x)} \geq \frac{m_1 \cdot x^n}{M_2 \cdot x^k} = \frac{m_1}{M_2} \cdot x^{n-k}.$$

Putting these estimates toegether we get the following OP-estimate for the rational fraction R :

$$\frac{m_1}{M_2} \cdot x^{n-k} \leq R(x) = \frac{P_1(x)}{P_2(x)} \leq \frac{M_1}{m_2} \cdot x^{n-k} \quad (x \in [R; +\infty)).$$

See this method as well at the class exercises.

6.1.1. Checking questions to the theory and its use

1. What is the OPU-estimate of a polynomial P (Order Preserving Upper estimate)?

2. What is the OPL-estimate of a polynomial P (Order Preserving Lower estimate)?

3. Using the OP estimates of the polynomials P, Q give the OP estimates for the rational fraction P/Q.

4. Give the OPU and OPL estimates for the polynomial

$$P(x) := x^5 + 3x^3 - 2x^2 + 7x + 21 \quad (x \in \mathbb{R}).$$

5. Give the OPU and the OPL estimates for

$$f(x) := \frac{2x^7 - 3x^4 + 5x^2 - x + 6}{x^2 + x + 1} \quad (x \in \mathbb{R}).$$

6. Give the OPL and the OPU estimates for the sequence

$$x_n := n^4 - 2n^3 - +7n^2 - n + 13 \quad (n \in \mathbb{N}).$$

7. Give the OPU and OPL estimates for

$$x_n := \frac{n^5 - 2n^4 + 3n^3 - 4n^2 + 5n + 111}{n^3 - 2n + 3} \quad (n \in \mathbb{N}).$$

8. Consider the function $f(x) := \dfrac{1}{x^2 + x + 1}$ $(x \in \mathbb{R})$. Give a positive number $K > 0$ so that:

$$|f(x) - 0| < \frac{1}{100}$$

holds, if $x > K$.

9. Let $f(x) := x^4 + 2x^2 + x - 5000$ $(x \in \mathbb{R})$ be a function. Give a $K > 0$, so that

$$f(x) > 80000$$

is ture for all numbers $x > K$.

10. Consider the function $f(x) := \dfrac{x^4 + 2x^3 - x + 12}{x^2 + x + 1}$ $(x \in \mathbb{R})$. Find a number $K > 0$ so that:

$$f(x) > 1000,$$

if $x > K$.

6.2. Exercises

6.2.1. Exercises for class work

OP estimates

1. Give an OPU estimate for the following polynomials:

(Give numbers $M > 0$ and $K > 0$ so that for all $x \geq K$ we have $P(x) \leq M \cdot x^n$.)

(a) $P(x) = 4x^5 - 3x^4 - 2x^2 - 5$;

(b) $P(x) = 2x^3 - 3x^2 + 6x + 7\,;$

(c) $P(x) = 6x^5 + 7x^4 + 10x^3 + x^2 + 2x + 3\,.$

2. Give an OPL estimate for the following polynomials:

(So: find numbers $m > 0$ and $k > 0$ so that for all $x \geq k$ we have $P(x) \geq m \cdot x^n$.)

(a) $P(x) = 6x^5 + 7x^4 + 10x^3 + x^2 + 2x + 3\,;$

(b) $P(x) = 2x^3 - 3x^2 + 6x + 7\,;$

(c) $P(x) = 4x^5 - 3x^4 - 2x^2 - 5\,.$

3. Give OPL and OPU estimates for the following fractions:

(a) $f(x) = \dfrac{3x^4 + 2x^3 + 5x^2 + 7x + 6}{5x^2 - 3x - 10}\,;$

(b) $f(x) = \dfrac{4x^3 - 10x^2 + 20x - 15}{7x^4 - 5x^3 - 10x^2 + 6x + 9}\,.$

4. Give OPL and OPU estimates for the following sequences:

(a) $a_n = 7n^3 - 4n^2 + 5n - 17 \;\; (n \in \mathbb{N}^+)\,;$

(b) $a_n = \dfrac{3n^4 + 7n^3 - 10n^2 - 13n + 6}{2n^5 - 8n^3 + 5n^2 + 9n - 7} \;\; (n \in \mathbb{N}^+)\,.$

5. Consider the function: $f(x) := \dfrac{x+1}{x^4 + x^2 + 1}$ $(x \in \mathbb{R})$. Find a number $K > 0$, so that

$$|f(x) - 0| < \frac{1}{1000},$$

if $x > K$.

6. Consider the function: $f(x) := x^3 - 2x^2 + 3 \;\; (x \in \mathbb{R})$ függvényt. Find a number $K > 0$, so that

$$f(x) > 200,$$

if $x > K$.

7. For the function $f(x) := \dfrac{x^3 - x^2 + 3x + 1}{x^2 - x + 1}$ $(x \in \mathbb{R})$ find a number $K > 0$, so that

$$f(x) > 100,$$

if $x > K$.

8. For the function $f(x) := \sqrt{x+1} - \sqrt{x} \quad (x \in [0;+\infty))$ find a number $K > 0$, so that

$$|f(x) - 0| < \frac{1}{1000},$$

if $x > K$.

6.2.2. Homework and exercises to practice

1. Give OPU and OPL estimates for the following polynomials:

(a) $P(x) = 7x^5 - x^4 - 2x^3 - x^2 - 6x - 10\,;$

(b) $P(x) = 12x^4 + 8x^3 + 3x^2 - 6x - 20\,;$

(c) $P(x) = x^5 + 9x^4 + 9x^3 + 10^2 + 11x + 33\,;$

(d) $P(x) = 4x^5 + 4x^4 + 2x^3 + 3x^2 + 10x + 5\,;$

(e) $P(x) = x^3 - 7x^2 - 6x + 20\,;$

(f) $P(x) = \frac{1}{10}x^5 - 99x^4 - 88x^3 - 67x^2 - 61x - 60\,.$

2. Give OPU and OPL estimates for the following fractions:

(a) $f(x) = \dfrac{x^5 + x^4 + 4x^3 + 7x^2 + x + 8}{3x^2 - 5x - 7}\,;$

(b) $f(x) = \dfrac{5x^3 - 9x^2 + 8x - 12}{4x^6 - 10x^5 + 3x^4 - 9x^3 - 11x^2 + 3x + 6}\,.$

3. Give OPU and OPL estimates for the following sequences:

(a) $a_n = n^3 - 7n^2 + 9n - 13 \;\; (n \in \mathbb{N}^+)\,;$

(b) $a_n = \dfrac{5n^4 + 3n^3 - 14n^2 - 9n + 7}{2n^5 + 11n^3 - 4n^2 + 5n - 17} \;\; (n \in \mathbb{N}^+)\,.$

4. For the function $f(x) := \dfrac{x^2 + x + 8}{x^3 + 2x^2 + 1} \quad (x \in (0;+\infty))$ find a nmumber $K > 0$ so that

$$|f(x) - 0| < \frac{1}{100}$$

holds, if $x > K$.

5. Consider the function:$f(x) := 2x^4 - 2x^3 + 3x^2 - x + 1 \quad (x \in \mathbb{R})$. Find a number $K > 0$ so that:

$$f(x) > 2018$$

holds, if $x > K$.

6. Consider the function $f(x) := \dfrac{x^4 - x + 13}{x^2 + x} \quad (x \in (0; +\infty))$. Find a number $K > 0$ so that:

$$f(x) > 1000$$

holds, if $x > K$.

7. For the function $f(x) := \sqrt{x^2 + 1} - x \quad (x \in \mathbb{R})$ find a number $K > 0$, so that

$$|f(x) - 0| < \frac{1}{100}$$

holds, if $x > K$.

7. Statements, logical operations

7.1. Theory

We are going to learn some basic notions like: logical statements, operations with statements (conjunction, disjunction, negation, implication, equivalence).

Logical statements

We consider the logical statements/arguments as basic notions, and we will denote them by letters like: $p, q, r \ldots$ or $A, B, C \ldots$. A logical statement always has a unique logical value: true or false. We say that a logical statement has a value of truth. This means that if we have a logical statement we can decide if it is true or false. For example:

1. $5 > 4$. Logical value: true. In other words: this statement is *true*.

2. $10 \geq 25$. This statement is *false*.

We also associate to a logical statement as its logical value a 0 if the statement is false, or a 1 if it is true.
Sometimes a statement is depending on one or more variables, which can take values from the so called *basic set* or *universe*. For example:

1. $P(x): \quad x + 3 \leq 5 \quad (x \in \mathbb{R})$,

2. $P(x; y): \quad x^2 + y^2 > 1 \quad (x \in \mathbb{R},\ y \in \mathbb{R})$.

These type of statements or sentences are called *open statements* or *predicates*, because their value of truth depends on the value(s) of the variable(s). For example the open statement

$$P(x): \quad x + 3 \leq 5$$

is true for $x = 1$, but is false if $x = 8$.

The set of all values of the possible variables for which the statement is true is called the *set of truth*.

For the previous open statement:

$$P(x): \quad x + 3 \leq 5 \quad (x \in \mathbb{R})$$

the variable x can take real values (so the basic set is $\mathbb{R}$) and the set of truth is the set of all real solutions of the inequality

$$x + 3 \leq 5,$$

which is $x \in (-\infty; 2]$.

Operations with logical statements, truth table

Using the following basic operations we can define new logical statements: *negation*: $\neg$, *conjunction*: $\wedge$, *disjunction* : $\vee$, *implication*: $\Longrightarrow$ and *equivalence*: $\Longleftrightarrow$. It is common to define the logical values of these statements using the truth table. In the columns we denote the statements that occur in the operation and in the rows we write all the possible logical values (1: true, 0: false) in all combinations.

p	q	$\neg$ p	p $\wedge$ q	p $\vee$ q	p $\Longrightarrow$ q	p $\Longleftrightarrow$ q
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

The truth table will be used to find the truth value of some composed statements or to prove the equality of two or more statements. See exercises at class work.

Quantifiers

Let us introduce the so called *universal quantifier*: $\forall$ and the *existential quantifier*: $\exists$. These two symbols are the logical *quantifiers*. Using these quantifiers we can define new statements from open sentences. For example if $P(x)$ is a statement depending on some variable denoted by x from a set T, then we can formulate new statements using our quantifiers in the following way:

$$\forall\ x \in T : P(x).$$

This means, that the statement $P(x)$ is true for all the values of x from the set T.

In a similar way the statement:

$$\exists\ x \in T : P(x)$$

means that there is at least one x value from the set T for which the statement $P(x)$ is true.

Examples:

1. $A(x) :\ \forall x \in \mathbb{R} : x^2 + 1 > 0$. This statement is obviously true.

2. $B(x) :\ \forall x \in \mathbb{R} : x + 3 \leq 5$. This is false, because for example if $x = 6$ then we get: $6 + 3 = 9$, what is not less or equal to 5.

3. $P(x) :\ \exists x \in \mathbb{R} : x + 3 \leq 5$. This is true, since for example if $x = 0$, then we get $P(0) :\ \ 3 + 0 = 3 \leq 5$. Here is enough to give only one value of x for which the statement is true, there is no need to find all the x values for which $P(x)$ is true.

4. $P(x) :\ \exists x \in [7, +\infty) : x + 3 \leq 5$. This statement is false, because whenever $x \geq 7$ we get $x + 3 \geq 10$.

5. $\exists x \in \mathbb{R} :\ x^2 - 3x + 2 = 0$.

   This is true since this quadratic equation has two roots, so $x = 1$ or $x = 2$ satisfies our statement.

6. $\exists\, x \in \mathbb{R}:\ -x^2 + x - 1 = 0.$

   This is false because the discriminant of the equation is $D = -3 < 0$, so we can conclude that there are no real roots.

7. $\forall\, x \in \mathbb{R}:\ -x^2 + x - 1 < 0.$

   This is true (sketch the graph of this parabola).

8. $P(x;y):\ \forall\,(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1.$

   This is false, since for example $(x,y) = (0,0)$ does not satisfy the given inequality.

9. $\forall\ y \in \mathbb{R} : x^2 + y^2 > 1.$

   This is an open statement with variable $x \in \mathbb{R}$. (It is open, because there is also a variable x which is influencing the inequality).

10. $\exists\ x \in \mathbb{R}\ \forall\, y \in \mathbb{R} : x^2 + y^2 > 1.$

    This statement is true, because for example if $x = 2$ then we have:

    $$4 + y^2 > 1,$$

    which holds for all $y \in \mathbb{R}$.

11. $\forall\ x \in [y,\, +\infty) : x + 3 \le 5 \qquad (y \in \mathbb{R}).$

    This is an open statement (open by the variable y).

12. $\exists\ y \in \mathbb{R}\ \forall\, x \in [y,\, +\infty) : x + 3 \le 5.$

    This statement is not open anymore and its logical value is false. To prove this let us take an arbitrary $y \in \mathbb{R}$.

    If $y \le 2$, then for example with $x := 3$ we have $3 \in [y,\, +\infty)$, but $3 + 3 \le 5$ is not true.

    When $y > 2$, then with $x := y + 1$ we get $y + 1 \in [y,\, +\infty)$, but $y + 1 + 3 \le 5$ is not true, so there is no such $y \in \mathbb{R}$.

13. $\forall\, \varepsilon > 0\ \ \exists\, \delta > 0 :\ 2\delta < \varepsilon.$

    This statement is true, because for a given arbitrary positive number ε any positive δ is good, wich satisfies the inequality

    $$\delta < \frac{\varepsilon}{2},$$

    so for example $\delta := \dfrac{\varepsilon}{3} \in (0; +\infty)$ will do.

14. $\exists\, K > 0\ \forall\, n \in \mathbb{N} :\ 2n < K.$

    This statement is false, because if we suppose that there is a positive number K with the upper property, then putting

$$n := \frac{[K]+1}{2} \in \mathbb{N}$$

    will fail to fulfill the condition above, since $2n = [K]+1 > K$. Here $[K]$ denotes the integer part of K.

Negation of expressions containing quantifiers

1. Consider statement 12 from above:

$$\exists\, y \in \mathbb{R}\ \forall\, x \in [y, +\infty) : \quad x + 3 \leq 5.$$

It's negation will be:

$$\forall\, y \in \mathbb{R}\ \exists\, x \in [y, +\infty) : \quad x + 3 > 5.$$

So what we have to do is: to replace the quantifiers with the opposite ones (instead of $\forall$ we put $\exists$ and conversely) and we take the negation of the last statement part. Formally:

$$\neg\, (\exists\, y \in \mathbb{R}\ \forall\, x \in [y, +\infty) : \quad x + 3 \leq 5) \ = \ \forall\, y \in \mathbb{R}\ \exists\, x \in [y, +\infty) : \quad x + 3 > 5.$$

2. Consider now the true statement in 13:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 :\ 2\delta < \varepsilon.$$

It's negation (of course is a false statement) is:

$$\exists\, \varepsilon > 0\ \forall\, \delta > 0 :\ 2\delta \geq \varepsilon.$$

„If-then" statements, implications Cosnider the open statements $A(x)$ and $B(x)$ with the variable x which takes values from the type-set denoted by T. In this case the:

„if $A(x)$ is true, then $B(x)$ is also true"

statement is called an *implication* and we denote it in a short way by:

$$A(x) \Longrightarrow B(x).$$

In other words, we can refer to an implication as one of the following statements:

- „Statement $A(x)$ implies statement $B(x)$."
- „From $A(x)$ we can conclude $B(x)$."
- „Statement $A(x)$ is a sufficient condition for $B(x)$ to be true."

- „Statement $A(x)$ is sufficient to $B(x)$."

  It is remarkable, that the sufficient condition $A(x)$ is at the left-hand side of the implication-arrow $\Longrightarrow$.

- „The condition $B(x)$ is a necessary condition for $A(x)$ to be true."

- „$B(x)$ is a necessary condition for $A(x)$."

  We can also emphasize that formally the necessary condition $B(x)$ is at the right-hand side of the implication-arrow $\Longrightarrow$.

- „For all values $x \in T$ when $A(x)$ is true, $B(x)$ is also true."

  We can shortly write this using the quantifier $\forall$:

$$\forall\ x \in T\ A(x) : B(x).$$

Remark: Using the $\forall$ quantifier to formalize an „if-then" statement it makes easier to understand the statement itself, the proofs, or the negation of the original statement.

Example:

Consider the implication $x \geq 3 \Longrightarrow x > 1$ for real numbers x. Some ways to formulate this statement are the following:

- If $x \geq 3$, then $x > 1$.

- From $x \geq 3$ we can conclude that $x > 1$.

- The condition $x \geq 3$ is sufficient for $x > 1$ to be true.

- $x \geq 3$ is a sufficient condition for $x > 1$.

- The condition $x > 1$ is necessary to the condtion $x \geq 3$ to be true.

- $x > 1$ is a necessary condition for $x \geq 3$.

- For all $x \in \mathbb{R}$ for which $x \geq 3$ is ture, $x > 1$ is also true, or using the quantifier:

$$\forall x \in \mathbb{R}\ \ x \geq 3 :\ x > 1.$$

Obviously the upper implication is true.

„If and only if " statements or equivalences

Let again $A(x)$ and $B(x)$ be two open statements depending on the variable $x \in T$. The statement „$B(x) \Longrightarrow A(x)$" is called the *reversal* of the statement „$A(x) \Longrightarrow B(x)$" . If the reversal of a true statement is also true, then we say that the original statement is *reversable.*

Examples:

Consider the last implication, which was true:

$$x \geq 3 \Longrightarrow x > 1.$$

Its reversed statement is:

$$x > 1 \Longrightarrow x \geq 3,$$

which can also be formulated in in the following way:

$$\forall\, x \in \mathbb{R}\ \ x > 1 :\ x \geq 3.$$

From this form we can conclude that this implication (so the reversal of the original statement) is false, since for example if $x = 2$ then $2 \in \mathbb{R}$ and $2 > 1$, but $2 \geq 3$ is not true. So the $x \geq 3 \Longrightarrow x > 1$ statement is not reversable.

When $A(x) \Longrightarrow B(x)$ and its reversed implication $B(x) \Longrightarrow A(x)$ is also true, then we say that:

„$A(x)$ *is equivalent* to $B(x)$".

This means that:

$$(\, A(x) \Longrightarrow B(x)\,) \quad \text{and} \quad (\, B(x) \Longrightarrow A(x)\,).$$

This new statement is called *equivalence* and we use the following notation for it:

$$A(x) \Longleftrightarrow B(x).$$

Other ways to fromulate this:

- „$A(x)$ and $B(x)$ are equivalent."
- „$A(x)$ is true *if and only if* $B(x)$ is true as well."
- „$A(x)$ is true *iff* $B(x)$ is true as well."
- „$A(x)$ is a *sufficient and necessary* condition for $B(x)$ to be true."
- „$A(x)$ is true *exactly when* $B(x)$ is true as well."

Since an equivalence consists of two (reversed) implications, and these last ones can be given by the „for all" quantifier, the equivalence:

$$A(x) \Longleftrightarrow B(x)$$

can also be given in the following form:

$$(\,\forall\, x \in T\ A(x) :\ B(x)\,) \quad \text{and} \quad (\,\forall\, x \in T\ B(x) :\ A(x)\,).$$

This last fromulation is very useful when we deal with the proofs of some theorems or exercises.

Examples:

1. If $x \in \mathbb{R}$ consider the following implication:

$$x \neq 0 \Longrightarrow x^2 > 0.$$

Reversing it we get the implication:

$$x^2 > 0 \Longrightarrow x \neq 0.$$

It can be proved, that both implications are true, so we have the following equivalence:

$$x \neq 0 \Longleftrightarrow x^2 > 0\ .$$

Some forms of this equivalence:

- $x \neq 0$ is equivalent to $x^2 > 0$.
- $x \neq 0$ if and only if $x^2 > 0$.
- $x \neq 0$ iff $x^2 > 0$.
- $x \neq 0$ is a sufficient and necessary condition for $x^2 > 0$ to be true.
- $x \neq 0$ is true exactly when $x^2 > 0$.

2. Consider the fixed numbers $a, b, c \in \mathbb{R},\ a \neq 0$ and the variable $x \in T := \mathbb{R}$.

We know that the following statement is *true*:

$$\exists\, x \in \mathbb{R}:\ ax^2 + bx + c = 0 \quad\Longleftrightarrow\quad D = b^2 - 4ac \geq 0.$$

7.1.1. Checking questions to the theory and its use

1. Consider the functions $f(x) := 1 - x^2\ (x \in \mathbb{R})$ and $g(x) := 2x\ (x \in \mathbb{R})$. Give a sufficient and necessary condition $A(x)$ so that the following statements to be true:

   (a) $f(x) = g(x) \quad\Longleftrightarrow\quad A(x)$;

   (b) $\forall\, x \in \mathbb{R} : f(x) \leq g(x) \quad\Longleftrightarrow\quad A(x)$;

   (c) $\forall\, x \in \mathbb{R} : f(x) \geq g(x) \quad\Longleftrightarrow\quad A(x)$.

2. Consider the following open statement, where $n \in \mathbb{N}$:

   $$A(n):\quad 3^n \geq n^2 + 2n + 1.$$

   Are the following statements true:

   (a) $A(0), A(1), A(2)$.

(b) $\exists\, N \in \mathbb{N}:\ A(N)$ is false.

(c) $\exists\, N \in \mathbb{N}\ \forall\, n \in \mathbb{N}\, n \geq N:\ A(n)$ is true.

3. Give the negation of statement (c), so negate the following statement:

$$\exists\, N \in \mathbb{N}\ \forall\, n \in \mathbb{N}\ \ n \geq N : 3^n \geq n^2 + 2n + 1.$$

4. Give the negation of the following statement, and decide if the statement or its negation is true or false:

$$\forall\, x \in \mathbb{R}\ \ \exists\, y \in [x; +\infty) : y - 5 < 21.$$

5. Give a sufficient and necessary condition on the real numbers $x, y \in \mathbb{R}$ so that the points $A(0;1), B(1;0), C(-1;0)$ and $D(x+y; x-y)$ form a square $ABCD$.

6. Give a sufficient and necessary condition on the real numbers $x, y \in \mathbb{R}$ so that the point $C(x;y)$ has the same distance from $A(-1;0)$ and from $B(1;2)$.

7. Give the mathematical/logical description of the following statements, using quantifiers, and decide if the statement is true or false:

   (a) The sum of the square of two real numbers equals 0, if and only if both numbers are 0;

   (b) One triangle is rectangular, iff the sum of the square of two proper sides equals the square of the third of its sides.

   (c) An integer can be deivided by 6 iff we can divide it by 2 and 3 as well.

   (d) One point $(x;y) \in \mathbb{R}^2$ in the plane is on the circle with centre $(0;0)$ and radius of 2 iff the sum of the squares of its coordinates equals 4.

   (e) If the square of a number is 4, then this number is 2.

   (f) The square of a number is 4, if its absolute value is 2.

   (g) The distance of a number from 1 is less then 3, if this number is at least -2 and at most 4.

   (h) The set of the natural numbers has a minimal value.

8. Consider the following open statements. Put in the place of the $\square$ one of the symbols : $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ so that the statement to become true (where we can write $\Longleftrightarrow$ put only this one there):

   (a) $x = \sqrt{16} \quad \square \quad x = 4;$

   (b) $4x^2 = 16 \quad \square \quad x = -2;$

   (c) $x^2 > 0 \quad \square \quad x \neq 0.$

9. Write down the following statement with mathematical and logical symbols and then decide if it is true or false. Give its negation as well.

   For almost all big natural numbers n we have:
$$\frac{n^4 - n^2}{n+1} > 4000.$$

10. Decide whether the following statement is true or false. Give its negation too.
$$\exists\, K > 0 \ \forall\, x \in (K; +\infty) : \ \frac{x^2 + x}{x^3 + 10} < \frac{1}{1000}.$$

11. Consider the functions: $f(x) := x^2 - 2x + 2 \ (x \in \mathbb{R})$ and $g(x) := a \ (x \in \mathbb{R})$, where a is a real parameter. Give sufficient and necessary conditions $A(a)$ so that the following statements to become true:
    (a) $\{x \in \mathbb{R} : f(x) = g(x)\} = \emptyset \iff A(a)$;
    (b) $\exists\,!\, x \in \mathbb{R} : f(x) = g(x) \iff A(a)$;
    (c) The set $A := \{x \in \mathbb{R} : f(x) = g(x)\}$ has exactly two elements $\iff A(a)$.

12. Consider the following open statement for natural numbers $n \in \mathbb{N}$:
$$A(n) : \quad \frac{n^4 + n + 1}{n^2 + 2} > 100.$$
    Are the following statements true or false:
    (a) $A(0); A(1); A(2)$.
    (b) $\exists\, N \in \mathbb{N} : \ A(N)$ is true.
    (c) $\exists\, N \in \mathbb{N} \ \forall\, n \in \mathbb{N}\, n \geq N : \ A(n)$ is false.

13. Is the following statement true:
$$\exists\, y \in \mathbb{R} \ \forall\, x \in (-\infty; y] : x - 5 < 21?$$

14. Let $a \in \mathbb{R}$ be a real parameter. Consider the following curves given by these two equations $x^2 + y^2 = 1$ and $y = x + a \ (x, y \in \mathbb{R})$. Give conditions $A(a)$ so that the following statements to be true:
    (a) The given curves have no common points $\iff A(a)$;
    (b) The given curves have exactly one common point $\iff A(a)$;
    (c) The given curves have exactly two common points $\iff A(a)$.

15. Is the following implication true (x denotes a real number):
$$|x - 1| < x \iff x > \frac{1}{2}.$$

16. Give the negation of the following statement and decide whether the statement or its negation is true or false:

$$\exists\, K > 0 \ \forall\, x \in (K; +\infty): \ \frac{x^3 + 2x^2 + x + 10}{x^2 + x + 1} > 2018.$$

17. Give the following statement in mathematical form:

    There exists a positive number K so that for all the positive numbers x greater then K we have: $\dfrac{\sqrt{x}}{x} < 100^{-1}$.

    Is this statement true?

18. Is the following statement true:

$$\exists\, A \subseteq \mathbb{R}: \ (|x^2 - 4| = 1 \iff x \in A)\,?$$

19. Is the following statement true, if x denotes a real number:

$$\exists\, A \subseteq \mathbb{R}: \ ((|x - 1| < 1/2 \ \wedge \ |x - 1/3| < 3/4|) \iff x \in A)\,?$$

20. Consider the following function:

$$f(x) := \frac{x^2 + 1}{x} \quad (x > 0).$$

    Is it true that:

$$\forall\, y \in [2; +\infty) \ \exists\, x \in (0; +\infty): f(x) = y\,?$$

21. Consider the following function:

$$f(x) := \cos x \quad (x \in (-\pi/2; +\pi/2)).$$

    Is the following implication true:

$$\text{If } x \in \mathbb{R} \text{ satisfies the condition } |2x| < 1 \implies f(x) > \frac{1}{\sqrt{2}}\,?$$

    What is the reversed statement? Is it true or false?

7.2. Exercises

7.2.1. Exercises for class work

Statements and quantifiers

1. Use the truth table to prove the following statements:

(a) $(A \wedge B) \vee C \;=\; (A \vee C) \wedge (B \vee C)$;

(b) $(A \vee B) \wedge C \;=\; (A \wedge C) \vee (B \wedge C)$;

(c) $\neg(\neg A) \;=\; A$;

(d) $\neg(A \wedge B) \;=\; \neg A \vee \neg B$;

(e) $\neg(A \vee B) \;=\; \neg A \wedge \neg B$;

(f) $A \Rightarrow B \;=\; \neg A \vee B$;

(g) $\neg A \Rightarrow \neg B \;=\; B \Rightarrow A$;

(h) $(\neg A \vee B) \Rightarrow B \;=\; A \vee B$;

2. Consider the open statements $A(x)$, $B(x)$, $C(x)$ and $D(x)$ on the universe $\mathbb{R}$ of real numbers:

$A(x)\colon x$ is a real positive number;

$B(x)\colon x$ is a real number for which we have: $x^2 + x - 6 = 0$;

$C(x)\colon x = -3$;

$D(x)\colon x = 2$.

Explain in different ways the following statements and check if they are true or false:

1. $C \Longrightarrow B$;	2. $C \Longrightarrow \neg A$;	3. $D \Longrightarrow A$;
4. $D \Longrightarrow B$;	5. $B \Longrightarrow (C \vee D)$;	6. $B \iff (C \vee D)$;
7. $(A \wedge B) \iff D$;	8. $\neg(\neg A \wedge D)$;	9. $(\neg A \wedge B) \iff C$.

3. Consider the following statements (all the variables are real numbers):

(a) $x = 0$ and $y = 0 \Longrightarrow x^2 + y^2 = 0$;

(b) $xy = xz \Longrightarrow y = z$;

(c) $x > y^2 \Longrightarrow x > 0$;

(d) $x^2 + y^2 = 12x + 16y - 75 \Longrightarrow 25 \le x^2 + y^2 \le 225$.

What are the reversed statements here? In each case decide which implication is true or false.

4. Consider the following open statements:

i) $n \geq 5 \ \ (n \in \mathbb{N})$;

ii) $\dfrac{1}{n+5} < 0,03 \ \ (n \in \mathbb{N})$;

iii) $x^2 + x - 1 = 0 \ \ (x \in \mathbb{R})$

For each of them solve the following exercises:

(a) Give a few values of the variables for which the statement is true.

(b) Give a few values of the variables for which the statement is false.

(c) Give all the values of the variables for which the statement is true.

(d) Formulate a new statement using the quantifier $\forall$. What is the logical value of this statement? Give its negation too.

(e) Formulate a new statement using the quantifier $\exists$. What is the logical value of this statemet? Give its negation too.

5. Consider the following statement:

$$\frac{1}{n} < 0,01 \ \ (n \in \mathbb{N}^+).$$

(a) Formulate a new statement using the quantifier $\forall$. What is the logical value of this statement! Give its negation too!

(b) Formulate a new statement using the quantifier$\exists$. What is the logical value of this statement! Give its negation too!

(c) Consider the following statement:

$$\forall\, n \geq N : \ \ \frac{1}{n} < 0,01 \ \ (N \in \mathbb{N}^+).$$

What type of statement is it?

(d) Consider the following statement:

$$\exists\, N \in \mathbb{N}^+ \ \forall\, n \geq N : \ \ \frac{1}{n} < 0,01.$$

Is it true or false? Give the negation of the statement too.

6. Formulate the following statements mathematically. Give their negations too, and decide which of them is true or false.

(a) There exists a natural number n so that: $\dfrac{n^2}{10n-7} > 100.$

(b) For all natural numbers n we have: $\dfrac{n^2}{10n-7} > 100.$

(c) There exists a natural number$N$ so that for all natural numbers n greater then N we have $\dfrac{n^2}{10n-7} > 100$.

Remarks: *The statement from point 3 we will also say in the following forms:*

a) For all great enough natural numbers n we have $\dfrac{n^2}{10n-7} > 100$.

b) The fraction $\dfrac{n^2}{10n-7}$ is greater then 100, whenever n is a great enough natural number.

7. Formulate the following statements mathematically. Give their negations too, and decide which of them is true or false.

(a) For all great enough natural numbers n we have:

$$n^4 - 35n^3 - 15n^2 + 13n + 10 > 2000.$$

(b) For all great enough natural numbers n we have:

$$\frac{2n^3+3}{n^5 - 3n^4 - 7n^3 + 2n^2 - 10n + 1} < 0,05.$$

(c) For all great enough natural numbers n we have:

$$\frac{2n^6 - 20n^5 + 3n^4 - 6n^3 + 3}{13n^3 + 100n^2 + 200} > 230.$$

8. Let a, b, x, y be real numbers. Are the following statements true or false:

(a) $a + b = 0 \Longleftrightarrow a^2 + b^2 = -2ab$?

(b) $a + b = 1 \Longleftrightarrow a^2 + b^2 = 1 - 2ab$?

(c) $x = -1 \Longleftrightarrow x^2 + x = 0$?

(d) $x = \sqrt{2} \Longleftrightarrow x^2 = 2$?

(e) $x^2 + y^2 - 2(x - y) = 7 \Longleftrightarrow$

$\Longleftrightarrow$ (x, y) is on the circle of center $(1; -1)$ and radius 3 ?

(f) Let a and b be non-negative real numbers. Is this statement true or false:

$$a^2 + b^2 = 0 \Longleftrightarrow \sqrt{a+b} = \sqrt{a} + \sqrt{b}\,?$$

(g) $x \geq 2 \Longleftrightarrow |1 - x| = x - 1$?

(h) $|x - 5| < 2 \Longleftrightarrow 3 < x < 7$?

(i) $y - x = |x| \Longleftrightarrow y = x \cdot (1 + \text{sign}(x))$?

(j) $\exists\ \log_{x^2-1/4}(1-x^2) \in \mathbb{R} \Longleftrightarrow \left[\dfrac{1}{|x|}\right] = 1$,

(where $[a]$ denotes the integer value/part of a)?

(k) $a = b \vee a = 3b \Longleftrightarrow a^3 - 3a^2b - ab^2 + 3b^3 = 0$?

(l) $x = 0 \vee x = \dfrac{\ln 3}{\ln 3 - \ln 2} \Longleftrightarrow 27^x - 3 \cdot 18^x - 12^x + 3 \cdot 8^x = 0$?

(m) $x = \pm\dfrac{\pi}{4} + k\pi\ \ (k \in \mathbb{Z}) \vee \operatorname{tg} x = 3 \Longleftrightarrow$

$$\Longleftrightarrow\ \sin^3 x - 3\sin^2 x \cos x - \sin x \cos^2 x + 3\cos^3 x = 0\ ?$$

9. Are the following statements true? Answer the given questions as well.

(a) $\forall x \in \mathbb{R} : (x-1)^2 + (x-5)^2 + (x-12)^2 \geq 62$ and we have equality here if and only if $x = 6$;

(b) Consider the function $f(x) := x + |x|\ \ (x \in [-1; 1])$. In this case:

a) $\forall\, x \in \mathbb{R} : 0 \leq f(x) \leq 2$;

b) $f(x) = 2$ iff $x = 1$;

c) f has a minimum value if and only if $x = 0$ or $x = -1$.

10. Consider the following function:

$$f(x) = |1 - |x||\ \ (x \in [-3; 2)).$$

Are the following statements true or false:

(a) $\forall\, x \in D_f : f(x) \geq 0$.

(b) $\forall\, x \in D_f : f(x) \leq 2$.

(c) $\exists!\, a \in D_f$ so that $\forall\, x \in D_f : f(a) \leq f(x)$.

(d) $\exists\, a \in D_f$ so that $\forall\, x \in D_f : f(a) \leq f(x)$.

(e) $\exists!\, a \in D_f$ so that $\forall\, x \in D_f : f(x) \leq f(a)$.

(f) $\exists!\, b \in D_f$ so that $f(b) = 1$.

(g) $\exists\, c \in D_f$ so that $f(c) = 0$.

(h) $\exists!\, x \in D_f$ so that $f(x) = x$.

(i) $\forall\, c \in \mathbb{R}$ the equation $f(x) = c$ has at least one solution.

(j) The equation $f(x) = c$ has at least one solution $\Longleftrightarrow c \in [0; 2]$.

(k) The equation $f(x) = c\ \ (c \in \mathbb{R})$ has exactly 4 solutions, iff $c \in (0; 1)$.

(l) The equation $f(x) = c\ \ (c \in \mathbb{R})$ has exactly 3 solutions, iff $c = 1$.

(m) The equation $f(x) = c$ $(c \in \mathbb{R})$ has exactly 2 solutions, iff $c = 0$.

(n) The equation $f(x) = c$ $(c \in \mathbb{R})$ has exactly 1 solution iff, $c \in (1; 2]$.

11. Are the following transformations/solutions true, if x denotes a real number:

(a) $\ln x^6 = 6 \iff 6 \cdot \ln x = 6 \iff \ln x = 1 \iff x = e$?

(b) $\sqrt{2x^2 - 2} > x$ $(x \in (-\infty; -1] \cup [1; +\infty)) \iff 2x^2 - 2 > x^2 \iff x^2 > 2 \iff x \in (-\infty; -\sqrt{2}) \cup (\sqrt{2}; +\infty)$?

(c) $x^2 + 2xy - 3y^2 = 0 \iff \left(\frac{x}{y}\right)^2 + 2 \cdot \frac{x}{y} - 3 = 0 \iff \left(\frac{x}{y} = 1 \vee \frac{x}{y} = -3\right) \iff$

$$\iff (y = x \vee x = -3y) \text{ ?}$$

(d) $x^2 + 2xy - 3y^2 = 0 \iff (x + y)^2 - 4y^2 = 0 \iff (x - y) \cdot (x + 3y) = 0 \iff$

$$\iff (y = x \vee x = -3y) \text{ ?}$$

(e) If $x, y \in \mathbb{Z}$, then:

$$x^2 + 2xy - 3y^2 = 5 \iff (x - y) \cdot (x + 3y) = 5 \iff (x; y) \in \{(2; 1); (-4; 1)\} \text{ ?}$$

(f) If $x, y \in \mathbb{Z}$, then :

$$x^2 + 2xy - 3y^2 = 3 \iff (x - y) \cdot (x + 3y) = 3 \iff (x; y) \in \emptyset \text{ ?}$$

12. For what statements A will be the following equivalences become true

(a) $\sqrt[4]{x} = x^4 \iff A(x)$?

(b) $\sqrt[3]{x} = x^3 \iff A(x)$?

(c) $\sqrt{1 - \cos x} = -2 \cdot \sin \frac{\pi}{4} \cdot \sin \frac{x}{2} \iff A(x)$?

(d) $\sqrt{\cos x} > 1 - \sin^2 x \iff A(x)$?

(e) $\cos x = x^2 - 4\pi \cdot x + 4\pi^2 + 1 \iff A(x)$?

(f) $\dfrac{x \cdot 2018^{1/x} + \frac{1}{x} \cdot 2018^x}{2} = 2018 \iff A(x)$?

(g) $\operatorname{tg}(x - y) = \operatorname{tg} x - \operatorname{tg} y \iff A(x; y)$?

(h) $x^2 + y^2 + 1 = xy + x + y \iff A(x; y)$?

(i) $max\{a, b\} = \dfrac{|a - b| + a + b}{2} \iff A(a; b)$?

(j) Consider the function $f(x) := \dfrac{x + 1}{x - 2}$ $(x \in \mathbb{R} \setminus \{2\})$. The equation $f(x) = y$ can be solved (for x) $\iff A(y)$?

7.2.2. Homework and exercises to practice

Statements and quantifiers

1. Use the truth table to prove the following statements:

(a) $A \wedge \neg(B \wedge A) \;=\; A \wedge \neg B$;

(b) $A \vee \neg(B \vee A) \;=\; A \vee \neg B$;

(c) $\neg((A \vee B) \wedge \neg A) \;=\; A \vee \neg B$;

(d) $A \Leftrightarrow B \;=\; (A \wedge B) \vee (\neg A \wedge \neg B)$;

(e) $((A \Rightarrow B) \wedge B) \Rightarrow A \;=\; A \vee \neg B$;

(f) $((A \Rightarrow B) \wedge A) \Rightarrow B \;=\; \neg A$;

2. Consider the following statements for the integers $n \in \mathbb{Z}$:

$A(n):$ n is divisible by 10;

$B(n):$ n can be divided by 5;

$C(n):$ n can be divided by 2.

Give the following statements, and decide if they are false or true:

(a) $A \Longrightarrow B$;

(b) $A \iff B \vee C$;

(c) $A \iff B \wedge C$;

(d) $A \Longrightarrow C$.

Check that in case (a) the reversed statement $B \Longrightarrow A$ is not true.

What can we say about the implication $C \Longrightarrow A$?

3. Formulate the following statements in different ways, using the structures *if-then*, $\Longrightarrow$, sufficient, necessary, quantifiers $\forall$, etc.

(a) The equality of two real numbers is a necessary condition, for their square to be equal too.

(b) The positivity of the discriminant of a quadratic equation is a sufficient condition for the equation to have real roots.

(c) The equality of the sides of a quadrangle is a necessary condition for this quadrangle to be a square.

(d) The equality of three line segments is a sufficient condition for these segments to be the sides of a triangle.

(e) The fact that a number is 5 and another one is 2 is a sufficient but not necessary condition for their sum to be 7.

4. Give a sufficient and necessary condition so that:

(a) the square of a real number to be positive.

(b) the quadratic expression $ax^2 + bx + c$ to be positive (non negative) (negative) (non positive) for all real numbers x ? a, b, c are fixed real numbers $(a \neq 0)$.

(c) the triangle ABC to be rectangular with hypotenuse c and other sides a, b?

(d) some convex quadrangle to be tangent to a circle?

(e) a given line segment to be seen under a rectangle from a point in the space?

(f) the quadratic equation $ax^2 + bx + c = 0$ to have two different real roots? a, b, c are fixed real numbers and $a \neq 0$.

(g) for the real number x to have a real number y so that $y^2 = x$?

5. All the variables in the following open statements are real numbers. Put in the place of the $\square$ one of the following symbols: $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ so that the statement to become true. (where we can put $\Longleftrightarrow$ put only this one):

(a) $x = \sqrt{4} \quad \square \quad x = 2;$

(b) $x^2 = 4 \quad \square \quad x = 2;$

(c) $x^2 > 0 \quad \square \quad x > 0;$

(d) $x^2 < 9 \quad \square \quad x < 3;$

(e) $x(x^2 + 1) = 0 \quad \square \quad x = 0;$

(f) $x(x + 3) < 0 \quad \square \quad x > -3\ .$

6. Consider the following open statement:

$$\frac{n^2}{2n+1} > 100 \quad (n \in \mathbb{N}).$$

(a) Give some values of the variable so that the statement becomes true.

(b) Give some values of the variable so that the statement becomes false.

(c) Give all the values of the variable when the statement is true!

(d) Make a new statement using the quantifier $\forall$. Is this statement true or false? Give the negation of this statement!

(e) Make a new statement using the quantifier $\exists$. Is this statement true or false? Give the negation of this statement!

7. Consider the following statements:

i) $\dfrac{1}{n+5} < 0,03 \quad (n \in \mathbb{N});$

ii) $\dfrac{n^2+5}{2n-1} > 213 \quad (n \in \mathbb{N});$

iii) $\dfrac{n^2}{2n+1} > 308 \quad (n \in \mathbb{N});$

iv) $\dfrac{73+10n-n^2}{2n+1} < -157 \quad (n \in \mathbb{N}).$

For each of them solve the following exercises:

(a) Make a new statement using the quantifier $\forall$. Is this statement true or false? Give the negation of this statement!

(b) Make a new statement using the quantifier $\exists$. Is this statement true or false? Give the negation of this statement!

(c) If $A(n)$ denotes the statement in point i), ..., iv) consider the following statement:

$$\forall n \geq N : \quad A(n) \qquad (N \in \mathbb{N}).$$

What kind of statement is it?

(d) If $A(n)$ denotes the statement in point i), ..., iv) consider the following statement:

$$\exists N \in \mathbb{N}^+ \ \forall n \geq N : \quad A(n).$$

Is this statement true or false? Give the negation of this statement!

8. Using the quantifiers give the mathematical/logical form of the following statements. Give their negation as well and decide, which one of them is true or false?

(a) There exists a natural number n so that : $\dfrac{n-2}{n^2-4n+2} < 0,07.$

(b) For all nautral numbers n we have : $\dfrac{n-2}{n^2-4n+2} < 0,07.$

(c) There is a natural number so that for all natural numbers n greater then that we have: $\dfrac{n-2}{n^2-4n+2} < 0,07.$

(d) For almost all great enough natural numbers n we have: $\dfrac{n-2}{n^2-4n+2} < 0,07.$

(e) For almost all great enough natural numbers n we have:

$$\frac{1}{2}n^3 - 25n^2 - 14n + 9 > 1000.$$

(f) For almost all great enough natural numbers n we have:

$$\frac{n^3+4n^2-3n+20}{3n^5-7n^4+4n^3-12n^2-12n+5} < 0,01.$$

(g) For almost all great enough natural numbers n we have:

$$\frac{2n^5-25n^4-17n^3+4n^2+5n-4}{18n^3+17n^2-16n+15} > 185.$$

9. Give the statement A so that the following equivalences to become true:

(a) $\left|\frac{1}{x}\right| = x^3 \iff A(x)$.

(b) $\frac{1}{\sqrt[3]{x}} = \sqrt{x^2} \iff A(x)$.

(c) $\sqrt{1+\cos x} = \cos\frac{x}{2} \iff A(x)$.

(d) $\sqrt{\sin x} = 1-\cos^2 x \iff A(x)$.

(e) $\sqrt{\sin x} > 1-\cos^2 x \iff A(x)$.

(f) $\sqrt{\cos x} < 1-\sin^2 x \iff A(x)$.

(g) $x^2+\sin x+\pi\cdot x+\frac{\pi^2}{4} = -1 \iff A(x)$.

(h) $\operatorname{tg}(x+y) = \operatorname{tg} x + \operatorname{tg} y \iff A(x;y)$.

(i) $\sqrt{(x-2)^2+(y+1)^2} < 2 \iff A(x;y)$. Illustrate the points of $A(x;y)$ in the geometric plane (2-dimensional coordinate system).

(j) $|x-3|+|y-2| < 1 \iff A(x;y)$. Illustrate the points of $A(x;y)$ in the geometric plane (2-dimensional coordinate system).

(k) $\min\{a,b\} = \frac{a+b-|a-b|}{2} \iff A(a;b)$.

(l) Consider the function $f(x) := \frac{3x+2}{x+1}$ $(x \in \mathbb{R}\setminus\{-1\})$. In this case:

The equation $f(x)=y$ can be solved (for x) $\iff A(y)$.

10. Are the following methods of solutions true:

(a) $\frac{\sin x}{x} + \frac{x}{\sin x} = 0 \iff \sin^2 x + x^2 = 0 \iff (\sin x = 0 \wedge x = 0) \iff x = 0$?

(b) $\sqrt{x-3}-\sqrt{2-x} > 0 \iff \sqrt{x-3} > \sqrt{2-x} \iff x-3 > 2-x \iff x > \frac{5}{2}$?

(c) $x^2-3xy+2y^2 = 0 \iff \left(\frac{x}{y}\right)^2 - 3\cdot\frac{x}{y}+2 = 0 \iff \left(\frac{x}{y} = 2 \vee \frac{x}{y} = 1\right) \iff$

$\iff (y = x \vee x = 2y)$?

(d) $x^2 - 3xy + 2y^2 = 0 \Longleftrightarrow (x-y)^2 + y^2 - xy = 0 \Longleftrightarrow$

$$\Longleftrightarrow (x-y)^2 - y\cdot(x-y) = 0 \Longleftrightarrow (x-y)\cdot(x-2y) = 0 \Longleftrightarrow (y = x \vee x = 2y) \; ?$$

(e) If $x, y \in \mathbb{Z}$, then:

$$x^2 + 3xy + 2y^2 = 2 \Longleftrightarrow (x+y)^2 + y\cdot(x+y) = 2 \Longleftrightarrow (x+y)\cdot(x+2y) = 2 \Longleftrightarrow$$

$$\Longleftrightarrow (x;y) \in \{(0;1);(3;-1);(0;-1);(-3;1)\} \; ?$$

11. Consider the function:

$$f(x) = \sqrt{1 - \sqrt{x}} \quad (x \in [0;1]).$$

Are the following statemnts true or false:

(a) $\forall\, x \in D_f : f(x) \geq 0$?
(b) $\forall\, x \in D_f : f(x) \leq 1$?
(c) $\exists!\, a \in D_f$ so that $\forall\, x \in D_f : f(a) \leq f(x)$?
(d) $\exists!\, a \in D_f$ so that $\forall\, x \in D_f : f(x) \leq f(a)$?
(e) $\exists\, c \in D_f$ so that $f(c) = 0$?
(f) $\exists\, x \in D_f$ so that $f(x) = \sqrt{x}$?
(g) $\forall\, x, t \in [1/8; 1/4] : \; |f(x) - f(t)| \leq |x - t|$?

12. Are the following statements true:

(a)
$$\exists\; x \in \mathbb{R}\; \forall\, y \in \mathbb{R} : (x + y = y)?$$

(b)
$$\forall\; y \in \mathbb{R}\; \exists\, x \in \mathbb{R} : (x + y = 0)?$$

(c)
$$\exists\; y \in \mathbb{R}\; \forall\, x \in \mathbb{R} : (xy^3 = -x)?$$

(d)
$$\forall\; x \in \mathbb{R}\; \exists\, y \in \mathbb{R} : (y^3 = x)?$$

(e)
$$\exists\; y \in \mathbb{R}\; \forall\, x \in \mathbb{R} : (x < y)?$$

(f)
$$\forall\; x \in \mathbb{R}\; \exists\, y \in \mathbb{R} : (x < y)?$$

8. Mathematical induction

8.1. Theoretic review

Theorem: *(The principle of mathematical (complete) induction)*

Let $A(n) \quad (n \in \mathbb{N})$ be statements defined on the natural numbers. If

1. $A(0)$ is true and
2. Whenever for an $n \in \mathbb{N}$ statement $A(n)$ true $\Longrightarrow \quad A(n+1)$ is true as well, then

$A(n)$ is true for all $n \in \mathbb{N}$.

Remarks:

1. The condition "$A(0)$ is true" is called the *first step* or the *starting step.*
2. The implication "Whenever for an $n \in \mathbb{N} : A(n)$ is true $\Longrightarrow \quad A(n+1)$ is true as well" is the inductional "step" form n to $n+1$.
3. In many exercises the "first" or starting statement is not for $n = 0$, but $n = 1$ or $n = 2$ or some specified $m \in \mathbb{N}$. The statement can be formulated in this form as well:

   Theorem: *(The principle of mathematical (complete) induction)*

   Let $A(n) \quad (n \in \mathbb{N})$ be statements defined on the natural numbers and $m \in \mathbb{N}$ fixed. If

   (a) $A(m)$ is true and
   (b) Whenever for an $m \leq n \in \mathbb{N}$ statement $A(n)$ true $\Longrightarrow \quad A(n+1)$ is true as well, then

   $A(n)$ is true for all $m \leq n \in \mathbb{N}$.

4. This principle of induction can also be extended to the set $\mathbb{Z}$ of inegers.

Binomial coefficients

Let $k, n \in \mathbb{N}$, $k \leq n$ and

$$0! := 1 \ , \qquad n! := \prod_{j=1}^{n} j \quad (1 \leq n) \ , \qquad \binom{n}{k} := \frac{n!}{k!(n-k)!} \ .$$

Some properties of the binomial coefficients:

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = \binom{n}{n-1} = n, \qquad \binom{n}{k} = \binom{n}{n-k},$$

And

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad (k = 1, \dots, n)\ .$$

Examples:

Equalities

1. Prove that for all $2 \leq n \in \mathbb{N}$ statement $A(n)$ is true, if:

$$A(n): \quad \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}.$$

Solution: First we chek it for $n = 2$:

$$A(2): \quad 1 - \frac{1}{2^2} = \frac{2+1}{2 \cdot 2} \iff \frac{3}{4} = \frac{3}{4}\ \surd.$$

Suppose that $A(n)$ is true for a natural number $2 \leq n \in \mathbb{N}$:

$$A(n): \quad \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n},$$

and we have to prove it for $n+1$, so:

$$A(n+1): \quad \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+2}{2n+2}.$$

Using the statement for n we get:

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{(n+1)^2}\right) =$$

$$= \underline{\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right)} \cdot \left(1 - \frac{1}{(n+1)^2}\right) = (A(n) \text{ is true}) =$$

$$= \underline{\frac{n+1}{2n}} \cdot \left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n} \cdot \frac{n^2+2n}{(n+1)^2} = \frac{n \cdot (n+2)}{2n \cdot (n+1)} = \frac{n+2}{2n+2}\ \surd.$$

So we proved that $A(2)$ is true and whenever $A(n)$ is true for a $2 \leq n \in \mathbb{N}$ then $A(n+1)$ is true as well. This means according to the complete induction principle, that $A(n)$ is true for all natural numbers n greater or equal to 2.

Remark: We can also prove this statement without induction in the following way (is a telescopic product):

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \prod_{k=2}^{n} \frac{(k-1) \cdot (k+1)}{k^2} =$$

$$= \frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{4 \cdot 6}{5^2} \cdots \frac{(n-1) \cdot (n+1)}{n^2} =$$

$$= \frac{1}{2} \cdot \frac{n+1}{n} = \frac{n+1}{2n}.$$

2. Prove that for all natural numbers $1 \leq n$ the following statement $A(n)$ is true:

$$A(n): \quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} \quad (1 \leq n \in \mathbb{N}).$$

Solution: Let's check it for the first allowed natural number, so for $n=1$:

$$A(1): \quad 1-\frac{1}{2}=\frac{1}{1+1} \iff \frac{1}{2}=\frac{1}{2} \ \surd.$$

Suppose that $A(n)$ is true for a fixed $1 \leq n \in \mathbb{N}$. Does it also imply $A(n+1)$ to be true?

$$A(n+1):$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2\cdot(n+1)}=\frac{1}{(n+1)+1}+\frac{1}{(n+1)+2}+\cdots+\frac{1}{2\cdot(n+1)} \iff$$

$$\iff 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}=\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+2}.$$

In the left hand side substitute the formula for $A(n)$ (which is supposed to be true):

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}\right)+\frac{1}{2n+1}-\frac{1}{2n+2}=\text{inductional supposing}=$$

$$=\left(\underline{\frac{1}{n+1}}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)+\frac{1}{2n+1}-\underline{\frac{1}{2n+2}}=$$

$$=\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+1}+\frac{1}{2n+2} \ \surd.$$

So according to the principle of mathematical induction we can conclude that $A(n)$ is true for all natural numbers $1 \leq n \in \mathbb{N}$.

Remark: The upper equality is called the *Catalan identity.*

Inequalities

1. Give all the natural numbers $n \in \mathbb{N}$ for which the following statement $A(n)$ is true:

$$A(n): \quad 2^n > n^2+4n+5.$$

Solution: Checking it for the first few natural numbers we find that $n=7$ is the first to be true:

$$A(7): \quad 2^7 > 7^2+4\cdot 7+5 \iff 128 > 82 \ \surd.$$

Suppose that $A(n)$ is true for a fixed natural number $7 \leq n \in \mathbb{N}$:

$$A(n): \quad 2^n > n^2+4n+5.$$

Using this let's prove it for $n+1$ so that $A(n+1)$ is true as well, so:

$$A(n+1): \quad 2^{n+1} > (n+1)^2 + 4\cdot(n+1) + 5.$$

Take the left-hand side of this inequality and use the inductional supposing:

$$2^{n+1} = 2^n \cdot 2 > (A(n) \text{ is true}) > 2\cdot(n^2+4n+5) = 2n^2+8n+10.$$

Looking at the statement we have to prove is *enough* to check, that:

$$2n^2+8n+10 > (n+1)^2 + 4\cdot(n+1) + 5.$$

This last inequality is:

$$2n^2+8n+10 > n^2+6n+10 \iff n^2+2n > 0,$$

which obviously holds if $n \geq 7$. "Putting toegether" all these estimates we get:

$$2^{n+1} > 2n^2+8n+10 > (n+1)^2+4\cdot(n+1)+5 \implies A(n+1) \ \surd.$$

This implies by induction that $A(n)$ is true for all $7 \leq n \in \mathbb{N}$.

2. Prove that for all $1 \leq n \in \mathbb{N}$ we have:

$$A(n): \quad \sum_{k=1}^{n} \frac{1}{k!} \leq \frac{5n-2}{2n}.$$

Solution: The first step is to check it for $n=1$:

$$A(1): \quad \frac{1}{1!} \leq \frac{3}{2} \iff 1 \leq \frac{3}{2} \ \surd.$$

Suppose that $A(n)$ is true for a natural number $1 \leq n \in \mathbb{N}$:

$$A(n): \quad \frac{1}{1!} + \frac{1}{2!} + \cdot + \frac{1}{n!} \leq \frac{5n-2}{2n}.$$

Using this, let's prove $A(n+1)$:

$$A(n+1): \quad \frac{1}{1!} + \frac{1}{2!} + \cdot + \frac{1}{n!} + \frac{1}{(n+1)!} \leq \frac{5n+3}{2n+2}.$$

Starting from the left-hand side of this inequality and using the inductional supposing we can estimate in the following way:

$$\underline{\left(\frac{1}{1!} + \frac{1}{2!} + \cdot + \frac{1}{n!}\right)} + \frac{1}{(n+1)!} \leq (\text{inductional supposing}) \leq \underline{\frac{5n-2}{2n}} + \frac{1}{(n+1)!}.$$

Comparing this with the right-hand side of $A(n+1)$ is *enough* to check, that:

$$\frac{5n-2}{2n} + \frac{1}{(n+1)!} \leq \frac{5n+3}{2n+2}.$$

Transforming this last inequality we get:

$$\frac{1}{(n+1)!} \leq \frac{5n+3}{2n+2} - \frac{5n-2}{2n} = \frac{4}{4n \cdot (n+1)} = \frac{1}{n \cdot (n+1)} \iff$$

$$\iff n \cdot (n+1) \leq (n+1)! \iff 1 \leq (n-1)!,$$

which is obviously true if $n \geq 1$. So we are ready to conclude by induction that $A(n)$ is true for all $1 \leq n \in \mathbb{N}$.

3. Prove that if $1 \leq n \in \mathbb{N}$ is a natural number and $x_1, x_2, \ldots, x_n \in (0; +\infty)$ with the following property:

$$x_1 \cdot x_2 \cdots x_n = 1,$$

then $A(n)$ is true:

$$A(n): \quad x_1 + x_2 + \cdots + x_n \geq n.$$

Solution: We prove this by induction on n. If $n = 1$, then we only have one positive number $x_1 > 0$ so that $x_1 = 1$. We have to prove for this x_1 that:

$$A(1): \quad x_1 \geq 1 \iff 1 \geq 1. \ \surd.$$

Suppose that $A(n)$ is true for some fixed natural number $1 \leq n \in \mathbb{N}$, which means in this case that if we choose n arbitrary positive real numbers, having their product 1 we will have their sum greater or equal to n. We prove using this, that $A(n+1)$ also holds, so if

$$x_1, x_2, \cdots, x_{n+1} > 0$$

and their product is 1, then:

$$A(n+1): \quad x_1 + x_2 + \cdots + x_{n+1} \geq n+1.$$

There are two possible cases. In the first case all the $n+1$ positive numbers are equal to 1. In this case $A(n+1)$ is also true, because this means that:

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{n+1 \text{ terms}} = n+1 \geq n+1,$$

which is true.

The second case is when not all the numbers are equal to 1. This implies that at least one of them must be less then 1. Let this number be x_n. In this case there also must be another number, which is greater then 1 (why?). Let this number be $x_{n+1} > 1$. So we have now:

$$x_1, x_2, \cdots, x_n, x_{n+1} > 0$$

and

$$x_1 \cdot x_2 \cdots x_{n+1} = 1 \ \wedge \ x_n < 1 \ \wedge \ x_{n+1} > 1.$$

We can use the inductional supposing for the following n numbers, whose product is obviously 1:

$$x_1, x_2, x_3, \cdots, x_{n-1}, x_n \cdot x_{n+1}.$$

For these numbers $A(n)$ is true, so we have:

$$x_1 + x_2 + \cdots + x_{n-1} + x_n \cdot x_{n+1} \geq n.$$

Taking the left-hand side of statement $A(n+1)$ and using the upper inequality we can use the following estimation:

$$x_1+x_2+\cdots+x_{n+1} = \underline{(x_1 + x_2 + \cdots + x_{n-1})}+x_n+x_{n+1} \geq \underline{n - x_n \cdot x_{n+1}}+x_n+x_{n+1}.$$

Comparing this to what we need is *enough* to prove that:

$$n - x_n \cdot x_{n+1} + x_n + x_{n+1} \geq n + 1.$$

This can be done as follows:

$$-x_n \cdot (x_{n+1} - 1) + x_{n+1} - 1 \geq 0 \iff (1 - x_n) \cdot (x_{n+1} - 1) \geq 0.$$

Using the previous conditions on x_n and x_{n+1}:

$$1 - x_n > 0 \;\wedge\; x_{n+1} - 1 > 0,$$

so their product is non-negative. We have proved $A(n+1)$. Using the induction principle $A(n)$ is true for all natural numbers $1 \leq n \in \mathbb{N}$.

Remark: We can also conclude from the upper proof that equality holds if and only if all the numbers are equal to 1. So, if the product of the positive numbers $x_1, x_2 \cdots x_n$ is 1, then:

$$x_1 + x_2 + \cdots + x_n = n \iff x_1 = x_2 = \cdots = x_n = 1.$$

Special case: Let $1 \leq n \in \mathbb{N}$ and $x_1, x_2, \cdots, x_n \in (0; +\infty)$ fixed positive arbitrary real numbers. Using the upper exercise for the following n positive numbers, with their product 1:

$$\frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}; \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}; \frac{x_3}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}; \cdots; \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}}.$$

we get the following inequality:

$$\frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} + \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} + \frac{x_3}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} + \cdots + \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} \geq n$$

and equality holds here iff:

$$\frac{x_1}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} = \frac{x_2}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} = \frac{x_3}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} = \cdots = \frac{x_n}{\sqrt[n]{x_1 \cdot x_2 \cdots x_n}} = 1 \iff$$

$$\iff x_1 = x_2 = \cdots = x_n = \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

With some rearrangements we get the following: for all $1 \le n \in \mathbb{N}$ and positive real numbers:

$$x_1, x_2, \cdots, x_n > 0$$

we have:

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 \cdot x_2 \cdots x_n},$$

and equality holds here if and only if:

$$x_1 = x_2 = \cdots = x_n.$$

It is easy to check, that if we choose these numbers to be *non-negative* (instead of beeing positive) the upper statements will still be true, so the following theorem holds:

Theorem: (The inequality of arithmetic and geometric means)

For all $1 \le n \in \mathbb{N}$ and $x_1, x_2, \cdots, x_n \in [0; +\infty)$ we have:

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

and equality holds here if and only if:

$$x_1 = x_2 = \cdots = x_n.$$

8.2. Exercises

8.2.1. Exercises for class work

Proving equalities

1. Prove that for all positive natural numbers $n \in \mathbb{N}^+$ we have:

(a) $\sum\limits_{k=1}^{n} k = \dfrac{n(n+1)}{2}$;

(b) $\sum\limits_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$;

(c) $\sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)} = \dfrac{n}{n+1}$;

(d) $\sum\limits_{k=1}^{n} \sin(kx) = \dfrac{\sin\dfrac{(n+1)x}{2} \cdot \sin\dfrac{nx}{2}}{\sin\dfrac{x}{2}}$ $(1 \le n \in \mathbb{N}; x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\})$.

2. (**Binomial theorem.**)

(a) Prove that for all $a, b \in \mathbb{R}$, $n \in \mathbb{N}^+$ we have:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \,.$$

(b) Using this theorem prove that:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \,; \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \qquad (n \in \mathbb{N}^+).$$

3. Prove that if $n \in \mathbb{N}^+$ and $a_1,\ q \in \mathbb{R}$, $q \neq 1$, then:

$$a_1 + a_1 q + a_1 q^2 + \ldots + a_1 q^{n-1} = a_1 \cdot \frac{q^n - 1}{q - 1}$$

(the sum of the first n terms of a geometric sequence).

Proving inequalities

4. Prove that:

(a) $2\sqrt{n+1} - 2 < \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1 \quad (n \in \mathbb{N}^+)$;

(b) $(2n)! < 2^{2n} \cdot (n!)^2 \quad (n \in \mathbb{N}^+)$;

(c) $\dfrac{1}{2\sqrt{n}} < \prod\limits_{k=1}^{n} \dfrac{2k-1}{2k} < \dfrac{1}{\sqrt{3n+1}} \quad (2 \leq n \in \mathbb{N})$;

(d) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \cdots \dfrac{1}{3n+1} > 1 \;\; (1 \leq n \in \mathbb{N})$;

(e) $|\sin(nx)| \leq n \cdot |\sin x| \;\; (\forall x \in \mathbb{R};\ n \in \mathbb{N})$.

5. (**The Bernoulli–inequality.**) Prove that, if $n \in \mathbb{N}^+$ and $-1 \leq h \in \mathbb{R}$, then

$$(1+h)^n \geq 1 + nh.$$

Consequence: For all $n \in \mathbb{N}^+$ and $h := \dfrac{1}{n}$ we get that:

$$\left(1 + \frac{1}{n}\right)^n \geq 2.$$

Sequences

6. Consider the sequence: $a_1 := \sqrt{2}$ and $a_{n+1} := \sqrt{2 \cdot a_n}$ $(n \in \mathbb{N}^+)$ defined by recursion. Prove that:

(a) $\forall\, n \in \mathbb{N}^+ :\ a_n < a_{n+1}$;

(b) $\forall\, n \in \mathbb{N}^+ : a_n < 2$.

Give an explicite formula for evaluating the $n-$th term of the sequence and prove it by induction. Using this fromula prove point (b) again.

7. Consider the sequence: $a_1 := \sqrt{2}$ and $a_{n+1} := \sqrt{2 + a_n}$ $(n \in \mathbb{N}^+)$ defined by recursion. Prove that:

(a) $\forall\, n \in \mathbb{N}^+ :\ a_n < a_{n+1}$;

(b) $\forall\, n \in \mathbb{N}^+ : a_n < 2$;

(c) $\forall\, n \in \mathbb{N}^+ : a_n = 2 \cdot \cos \dfrac{\pi}{2^{n+1}}$.

Prove the implication (c) $\Longrightarrow$ (b).

8. Consider the following sequence defined by recursion:

$$a_1 := \frac{1}{2};\ a_2 := \frac{1}{3}\ \wedge\ a_{n+2} := \frac{a_n \cdot a_{n+1}}{3a_n - 2a_{n+1}}\quad (n \in \mathbb{N}^+).$$

Prove that:

$$a_n = \frac{1}{1 + 2^{n-1}}\quad (n \in \mathbb{N}^+).$$

Other types

9. Prove that for all natural numbers $n \in \mathbb{N}$ the following number can be divided by 23:

$$2^{7n+3} + 3^{2n+1} \cdot 5^{4n+1}.$$

10. Prove that for all natural numbers $n \in \mathbb{N}$ if $x = (2 + \sqrt{3})^n$, then $\dfrac{x^2 + 2x - 3}{12}$ is a perfect square.

11. Let $a \in \mathbb{R} \setminus \{0\}$ be a number so that: $a + \dfrac{1}{a} \in \mathbb{Z}$. Prove that in this case we have:

$$a^n + \frac{1}{a^n} \in \mathbb{Z}\quad (\forall\, n \in \mathbb{N}).$$

8.2.2. Homework and exercises to practice

Proving equalities

1. Prove that for all positive natural numbers $n \in \mathbb{N}^+$ we have:

(a) $\sum\limits_{k=1}^{n}(-1)^{k-1} \cdot k^2 = (-1)^{n-1} \cdot \dfrac{n(n+1)}{2}$;

(b) $\sum\limits_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$;

(c) $\sum\limits_{k=1}^{n} k(3k+1) = n(n+1)^2$;

(d) $\sum\limits_{k=1}^{n} k(k+1) = \dfrac{n(n+1)(n+2)}{3}$;

(e) $\sum\limits_{k=1}^{n} k(k+1)(k+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$;

(f) $\sum\limits_{k=1}^{n}(2k-1) = n^2$;

(g) $\sum\limits_{k=1}^{n} k \cdot k! = (n+1)! - 1$;

(h) $\prod\limits_{k=1}^{n} \left(1 + \dfrac{1}{k}\right) = n+1$;

(i) $\sum\limits_{k=1}^{n} \dfrac{1}{(2k-1)2k} = \sum\limits_{k=1}^{n} \dfrac{1}{n+k}$;

(j) $$\sum_{k=0}^{n} \cos(kx) = \frac{\sin\dfrac{(n+1)x}{2} \cdot \cos\dfrac{nx}{2}}{\sin\dfrac{x}{2}} \quad (1 \le n \in \mathbb{N};\, x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}).$$

Proving inequalities

2. Prove that:

(a) $2^n > n^2 \quad (4 < n \in \mathbb{N})$;

(b) $3^n > n^3 \quad (4 \le n \in \mathbb{N})$;

(c) $\sum\limits_{k=1}^{n} \dfrac{1}{k^2} < \dfrac{2n-1}{n} \quad (2 \le n \in \mathbb{N})$;

(d) $\sum\limits_{k=1}^{n} \dfrac{1}{n+k} > \dfrac{1}{2} \quad (2 \le n \in \mathbb{N})$;

(e) $\sum\limits_{k=1}^{n} \dfrac{1}{n+k} > \dfrac{13}{24} \quad (2 \le n \in \mathbb{N})$;

(f) $\dfrac{n}{2} < \sum\limits_{k=1}^{2^n-1} \dfrac{1}{k} < n \quad (2 \le n \in \mathbb{N});$

(g) $\sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{k}} > \sqrt{n} \quad (2 \le n \in \mathbb{N});$

(h) $\prod\limits_{k=1}^{n} (2k)! > ((n+1)!)^n \quad (2 \le n \in \mathbb{N});$

(i) $2^n > 1 + n \cdot \sqrt{2^{n-1}} \quad (2 \le n \in \mathbb{N}).$

3. (**Generalized inequality of Bernoulli**).

Prove that if $n \in \mathbb{N}^+$; $x_1, \dots, x_n \in [-1, +\infty)$, and $x_1, \dots, x_n$ have all the same sign, then:

$$(1 + x_1) \cdot (1 + x_2) \cdot \ldots \cdot (1 + x_n) \ge 1 + x_1 + x_2 + \ldots + x_n \; .$$

4. Prove the following inequalities:

(a) $n! < \left(\dfrac{n+1}{2}\right)^n \quad (2 \le n \in \mathbb{N});$

(b) $\prod\limits_{k=1}^{n} \left(k + \dfrac{1}{2}\right) < \left(\dfrac{n+1}{2}\right)^{n+1} \quad (3 \le n \in \mathbb{N}).$

Sequences

5. Consider the sequence: $a_1 := \sqrt{\dfrac{1}{2}}$ and $a_{n+1} := \sqrt{\dfrac{1}{2} \cdot a_n}$ $(n \in \mathbb{N}^+)$. Prove that:

(a) $\forall\, n \in \mathbb{N}^+ : \; a_n > a_{n+1};$

(b) $\forall\, n \in \mathbb{N}^+ : \dfrac{1}{2} < a_n \le \dfrac{1}{\sqrt{2}}.$

Give an explicit formula for the $n-$th term of the sequence, and prove it by induction.

6. Consider the sequence: $a_1 := \dfrac{1}{\sqrt{2}}$ and $a_{n+1} := \sqrt{\dfrac{1}{2} + a_n}$ $(n \in \mathbb{N}^+)$. Prove that:

(a) $\forall\, n \in \mathbb{N}^+ : \; a_n < a_{n+1};$

(b) $\forall\, n \in \mathbb{N}^+ : a_n < \dfrac{1 + \sqrt{3}}{2}.$

Other types

7. Prove that for all natural numbers $1 \leq n \in \mathbb{N}$ the following number can be divided by 9:

$$4^n + 15n - 1.$$

9. Complex numbers

9.1. Theory

9.1.1. The notion of a complex number

When solving quadratic equations we met the situation when the discriminant was negative, so we had to take the square root of a negative number, which cannot be made in the "world" of the real numbers. In these cases we took the conclusion, that the equation had no real roots. For example there is no real number x so that:

$$x^2 + 1 = 0,$$

or in other words: there is no $x \in \mathbb{R}$ for which we have:

$$x^2 = -1.$$

Let's make one brave step and define a "new number" denoted by i and calling it the *imaginary unit* with the following property:

$$i^2 = -1.$$

It is obvious that $i \notin \mathbb{R}$. Using this newly introduced "object" we can define the so called *complex numbers.*

Definition: Let $x, y \in \mathbb{R}$ arbitrary real numbers and i the imaginary unit, so $i^2 = -1$. We will call the expressions

$$z := x + i \cdot y \quad (x, y \in \mathbb{R})$$

complex (composed) numbers. We will use $z, w, \varepsilon, ...$ or $z_1; z_2, w_1; ...$ to denote complex numbers.

For example:

$$z = 1 + 2 \cdot i; \;\; w = -3 - \sqrt{2} \cdot i; \;\; \varepsilon = \frac{2}{3} - i;$$

$$z_1 = 2 + 0 \cdot i = 2; \;\; z_2 = 0 - 6 \cdot i = -6 \cdot i; \ldots$$

If

$$z = x + i \cdot y$$

is a complex number, then we call x the *real part* of z and y the *imaginary part* of z. In notation:

$$z = x + i \cdot y \Longleftrightarrow x = \mathrm{Re}(z) \;\; \wedge \;\; y = \mathrm{Im}(z).$$

In the upper examples:

$$\mathrm{Re}(1+2\cdot i)=1 \ \wedge \ \mathrm{Im}(1+2\cdot i)=2;$$
$$\mathrm{Re}(w)=\mathrm{Re}(-3-\sqrt{2}\cdot i)=-3 \ \wedge \ \mathrm{Im}(w)=\mathrm{Im}(-3-\sqrt{2}\cdot i)=-\sqrt{2};$$
$$\mathrm{Re}(\varepsilon)=\mathrm{Re}\left(\frac{2}{3}-i\right)=\frac{2}{3} \ \wedge \ \mathrm{Im}(\varepsilon)=\mathrm{Im}\left(\frac{2}{3}-i\right)=-1;$$
$$\mathrm{Re}(z_1)=\mathrm{Re}(2+0\cdot i)=2 \ \wedge \ \mathrm{Im}(z_1)=\mathrm{Im}(2+0\cdot i)=0.$$

We can see here that the imaginary part is 0, so $z = x + 0 \cdot i = x \quad (x \in \mathbb{R})$ is a "pure" real number. So we can conclude that all the real numbers are also complex numbers with their imaginary part beeing 0.

$$\mathrm{Re}(z_2)=\mathrm{Re}(0-6\cdot i)=0 \ \wedge \ \mathrm{Im}(z_2)=\mathrm{Im}(0-6\cdot i)=-6.$$

In this case the real part is 0. These complex numbers of form:

$$z = 0 + y \cdot i = yi \quad (y \in \mathbb{R})$$

will be called "pure" *imaginary numbers*. Let us denote by $\mathbb{C}$ the set of all complex numbers, so:

$$\mathbb{C} := \{z = x + i \cdot y \ \big| \ x, y \in \mathbb{R}\}.$$

A more "precise" algebraic definition of the complex numbers will be done later in the next semester. Our first aim in this material is to be able to make some basic algerbraic calculations with complex numbers, so we introduce them in this quite simple way. So now we can also state that:

$$\mathbb{R} \subset \mathbb{C},$$

but $\mathbb{C}$ is a larger set then $\mathbb{R}$, since for example $i \in \mathbb{C} \setminus \mathbb{R}$.

Two complex numbers are considered to be equal if and only if their real and imaginary parts are equal respectively. So, if $z = x + iy$ and $w = a + ib \quad (x, y, a, b \in \mathbb{R})$ are complex numbers, then:

$$z = w \iff x + iy = a + ib \iff (x = a \ \wedge \ y = b).$$

This gives us the possibility to find a unique correspondence between the complex numbers of form $z = x+iy \in \mathbb{C}$ and ordered pairs $(x; y) \in \mathbb{R}^2$ (points of the two-dimensional plane). These last points may be familiar to the reader from geometry as 2-dimensional vectors or points in the cartesian coordinate system of the plane. Having this correspondence we can use many properties of the vectors or the ordered pairs in case of the complex numbers as well. For example we can represent them in a plane, the so called *Gauss-plane* of the complex numbers.

In the upper examples we have the following correspondences:

$$z = 1 + 2 \cdot i \longleftrightarrow P = (1; 2);$$
$$w = -3 - \sqrt{2} \cdot i \longleftrightarrow Q = (-3; -\sqrt{2});$$
$$\varepsilon = \frac{2}{3} - i \longleftrightarrow A = (2/3; -1);$$
$$z_1 = 2 + 0 \cdot i = 2 \longleftrightarrow B = (2; 0);$$
$$z_2 = 0 - 6 \cdot i = -6 \cdot i \longleftrightarrow S = (0; -6).$$

9.1.2. Representing complex numbers in the Gauss-plane

Consider the complex number $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$. Represent the real numbers x and y on two perpendicular axis of a coordinate system, where the horizontal axis (where x takes all the possible real values) is called the *real axe* and the vertical line (where we represent the y values) is called the *imaginary axis*. These two perpendicular axes intersect each other in point $(0; 0)$, the so called origin. As on the usual real line we also indicate the positive direction on both axes. This "new" coordinate system is called the Gauss-complex plane, and each complex point $z = x + iy \in \mathbb{C}$ corresponds to one and only one point $(x; y) \in \mathbb{R}^2$ in the cartesian two-dimensional plane denoted by $\mathbb{R}^2$.

As for the vectors we can also introduce the *length/absolute value* or *modulus* of a complex number. It is useful to define the *conjugate* of a complex number as well as follows:

Definition: Let $z = x + iy \ \ (x, y \in \mathbb{R})$ be an arbitrary complex number. Its *absolute value* or *length* or *modulus* is the nonnegative real number:

$$|z| := \sqrt{x^2 + y^2} \in [0; +\infty),$$

and its *conjugate* is the complex number:

$$\overline{z} := x - iy \in \mathbb{C}.$$

For example:

$$|7 - 4i| = \sqrt{49 + 16} = \sqrt{65};$$
$$|-1 + 2i| = \sqrt{1 + 4} = \sqrt{5};$$
$$|i| = \sqrt{1} = 1;$$
$$\overline{1 - 2i} = 1 + 2i;$$
$$\overline{-1 + 2i} = -1 - 2i;$$
$$\overline{-5i} = 5i.$$

Remark: Using the so called *polar coordinates* of a point in the geometric plane $\mathbb{R}^2$ we can also introduce the *trigonometric form* of a complex number, but as we mentioned before, we only need algebraic computations and some properties of the complex numbers, so the trigonometric representation and its use will come later in the next semester. We will only use the so called *algebraic form*: $z = x + iy \in \mathbb{C}$ of a complex number.

9.1.3. Operations with complex numbers

Definition: Let us introduce the four basic algebraic operations (addition, substraction, multiplication, division) in $\mathbb{C}$. Consider for this the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \ \ (x_1, x_2, y_1, y_2 \in \mathbb{R})$. We have by definition the following:

$$z_1 + z_2 := (x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + (y_1 + y_2) \cdot i;$$

$$z_1 - z_2 := (x_1 + iy_1) - (x_2 + iy_2) := (x_1 - x_2) + (y_1 - y_2) \cdot i;$$

$$z_1 \cdot z_2 := (x_1 + iy_1) \cdot (x_2 + iy_2) := (x_1 \cdot x_2 - y_1 \cdot y_2) + (x_1 \cdot y_2 + x_2 \cdot y_1) \cdot i;$$

$$z_1/z_2 := \frac{z_1}{z_2} := \frac{x_1 + iy_1}{x_2 + iy_2} := \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{x_2^2 + y_2^2} + \frac{-x_1 \cdot y_2 + x_2 \cdot y_1}{x_2^2 + y_2^2} \cdot i.$$

The results of the upper operations (the new complex numbers) are given in their algebraic form (we can read their real and imaginary parts). Think of it for a minute why exactly these definitions are the ones we want to use?

For example:

If $z_1 := 2 + 3i$ and $z_2 := -1 + 5i$, then we have:

$$z_1 + z_2 = (2 + 3i) + (-1 + 5i) = 1 + 8i;$$

$$z_1 - z_2 = (2 + 3i) - (-1 + 5i) = 2 + 3i + 1 - 5i = 3 - 2i;$$

$$z_1 \cdot z_2 = (2 + 3i) \cdot (-1 + 5i) = -2 + 10i - 3i + 15i^2 = -2 + 7i - 15 = -17 + 7i;$$

$$z_1/z_2 = \frac{z_1}{z_2} = \frac{2 + 3i}{-1 + 5i} = \frac{(2 + 3i) \cdot (-1 - 5i)}{(-1 + 5i) \cdot (-1 - 5i)} = \frac{13 - 13i}{1 - 25i^2} = \frac{13 - 13i}{26} = \frac{1}{2} - \frac{1}{2} \cdot i;$$

$$\frac{1}{1 + i} = \frac{1 - i}{(1 + i) \cdot (1 - i)} = \frac{1 - i}{1 - i^2} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2} \cdot i;$$

$$3z_1 := 3 \cdot z_1 = 3 \cdot (2 + 3i) = 6 + 9i;$$

$$iz_1 := i \cdot z_1 = i \cdot (2 + 3i) = 2i + 3i^2 = -3 + 2i;$$

$$z_1^2 = (2 + 3i)^2 = 4 + 12i + 9i^2 = -5 + 12i;$$

$$2z_1 = 2 \cdot (2 + 3i) = 4 + 6i;$$

$$\frac{1}{z_1 + 2z_2} = \frac{1}{(2 + 3i) + 2 \cdot (-1 + 5i)} = \frac{1}{13i} = \frac{i}{13i^2} = -\frac{1}{13} \cdot i.$$

Remark: We can observe, that the addition, substraction and multiplication with a real scalar of complex numbers is very similar to the correspondent operations in the world of (two dimnsional) plane-vectors, including the representations of these vectors as well. There is no such analogy for the multiplication and the division of complex numbers and the scalar or vectorial product (cross-product) of two vectors. We don't even speak of dividing two vectors.

If $z = x + iy \ (x, y \in \mathbb{R})$ is a complex number, then we have the following (in many cases useful) identity:

$$z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 - y^2 \cdot i^2 = x^2 + y^2 = |z|^2 \ (z \in \mathbb{C}),$$

so for any complex number z we have:

$$z \cdot \overline{z} = |z|^2 \ (z \in \mathbb{C}),$$

or

$$|z| = \sqrt{z \cdot \overline{z}} \ (z \in \mathbb{C}).$$

If $z = x + 0i = x \in \mathbb{R}$ is a real number, then $\overline{z} = x$ and we get back the well-known property of the real numbers:

$$|z| = |x| = \sqrt{z \cdot \overline{z}} = \sqrt{x \cdot x} = \sqrt{x^2},$$

or

$$\sqrt{x^2} = |x| \ \ (x \in \mathbb{R}).$$

Questions:

1. What is the geometric meaning of the following nonnegative real number, if z_1 and z_2 are two arbitrary complex numbers:

$$|z_1 - z_2|\ ?$$

2. Where are in the Gauss-plane all the complex numbers z for which we have:

$$|z - i - 1| < 1?$$

3. Where are in the Gauss-plane all the complex numbers z for which we have:

$$|z - i| = 2?$$

4. Where are in the Gauss-plane all the complex numbers z for which we have:

$$\frac{1}{|2z + 1/i - 2|} \leq 4?$$

9.1.4. Complex roots of polynomials with real coefficients

As a last point let us return to the initial problem of solving quadratic equations with negative discriminant and extend this question for some (easy-solvable) equations of higher degree, with real coefficients. For example solve the following quadratic equation (in this case with $x \in \mathbb{C}$):

$$x^2 + 2x + 4 = 0.$$

Using the well-known formula we get:

$$x_{1,2} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{-3}.$$

Since the discriminant is -3 we still don't have real solutions, but now on $\mathbb{C}$ what should we mean by $\sqrt{-3}$?

Now that we have introduced the imaginary unit i, for which we have $i^2 = -1$ we can make the following transformations (complete square form):

$$(x+1)^2+3 = 0 \iff (x+1)^2 = -3 \iff (x+1)^2 = 3 \cdot i^2 \iff (x+1)^2-(\sqrt{3} \cdot i)^2 = 0 \iff$$

$$\iff (x + 1 - \sqrt{3} \cdot i) \cdot (x + 1 + \sqrt{3} \cdot i) = 0.$$

From here we can see the two complex solutions:

$$x_{1,2} = -1 \pm \sqrt{3} \cdot i.$$

We can see that the two complex solutions are the conjugate of each other. Comparing our result to the numbers we have got from the solving formula, we can answer what $\sqrt{-3}$ should be, since :

$$\sqrt{-1} = i$$

and

$$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1} = \sqrt{3} \cdot i.$$

Remarks:

1. As you will see later (the next semester) the "square root" (or more generally the $n-$th ($2 \leq n \in \mathbb{N}$) root) of a complex number will have *two* (in the general case n) values, but now we make the following agreement: when $x \in (0; +\infty)$ is a positive real number, then

   $$\sqrt{-x} := \sqrt{x} \cdot i.$$

   Using this we can extend and use the quadratic equation's solving formula to all the cases when the discriminant is neagtive.

2. For example we will see later, that the complex square root of -1 is:

   $$\sqrt{-1} = \pm i,$$

   since

   $$(\pm i)^2 = i^2 = -1.$$

Let $a, b, c \in \mathbb{R}$ be arbitrary real numbers with $a \neq 0$ and consider the general quadratic equation:

$$ax^2 + bx + c = 0 \ \ (x \in \mathbb{C}),$$

where x can take complex values now, and suppose that the discriminant is negative (the other cases are the same, as we saw them in our preliminary studies at school):

$$b^2 - 4ac < 0.$$

With the upper agreement, we get as complex solutions the following ones:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2} \cdot \sqrt{-1}}{2a} = \frac{-b \pm i \cdot \sqrt{4ac - b^2}}{2a}.$$

Remark:

1. It can be proved, that the higher order polynomials P (for now only with real coefficients) can be factorized on $\mathbb{R}$ to the product of only linear or quadratic polynomials

with real coefficients. If we have this factorization (generally it is not easy to do it), then we can find all the real and complex solutions of the equation:

$$P(x) = 0 \quad (x \in \mathbb{C}).$$

For example solve on $\mathbb{C}$ the following equation (so find those numbers $x \in \mathbb{C}$ for which $P(x) = 0$):

$$x^4 - x^3 - 8x^2 + 9x - 9 = 0 \quad (x \in \mathbb{C}).$$

Solution: Observe, that (sometimes is not easy...):

$$x^4 - x^3 - 8x^2 + 9x - 9 = 0 \iff x^4 - 9x^2 - x^3 + 9x + x^2 - 9 = 0 \iff$$

$$\iff x^2 \cdot (x^2 - 9) - x \cdot (x^2 - 9) + (x^2 - 9) = 0 \iff (x^2 - 9) \cdot (x^2 - x + 1) = 0 \iff$$

$$\iff (x-3) \cdot (x+3) \cdot (x^2 - x + 1) = 0 \iff$$

$$\iff x - 3 = 0 \ \vee \ x + 3 = 0 \ \vee \ x^2 - x + 1 = 0.$$

We get the solutions:

$$x_{1,2} = \pm 3; \quad x_{3,4} = \frac{1 \pm \sqrt{3} \cdot i}{2}.$$

9.1.5. Checking questions to the theory and its use

1. When can we say that $z = w$, if $z, w \in \mathbb{C}$ are complex numbers?
2. What is the definition of $\mathbb{C}$?
3. What is the definition of the conjugate of a complex number z?
4. Give the real and imaginary parts of the complex number $\dfrac{2i - 1}{5 + 3i}$.
5. If $z = x + iy \in \mathbb{C}; x, y \in \mathbb{R}$ is a complex number, then give the real and imaginary parts of z^2.
6. Consider the following complex numbers: $z_1 \neq 0$ and $z_2 \in \mathbb{C}$. Give the real and imaginary parts of $\dfrac{z_2}{z_1}$.
7. For what complex numbers z do we have:

$$z^2 + \overline{z}^2 = 0\,?$$

8. Illustrate all the complex numbers z in the complex plane for which:

$$\mathrm{Re}(z - 1 + i) = 2\,.$$

9. Give all the complex numbers z in the complex plane for which:
$$-1 < \operatorname{Re} z \leq 3\,?$$
10. Evaluate $|z|$ for $z = (\sqrt{2} - i \cdot \sqrt[4]{3})^2$.
11. Where are the complex numbers z in the complex plane, for which z^3 is a real number?
12. Evaluate $\left(\dfrac{1-i}{1+i}\right)^{2018}$.
13. Evaluate $1 + \dfrac{1+i}{1-i} \cdot i$.
14. Solve the following equation in $\mathbb{C}$:
$$x^3 - 1 = 0.$$
15. What are the real and imaginary parts of $\dfrac{1}{z}$, if $z = x + iy \in \mathbb{C} \setminus \{0\}$.
16. Define the absolute value of a complex number z.
17. When is it true that: $z = \overline{z}\,?$
18. What is the geometric meaning of $|z - \sqrt{3} + i|$?
19. Evaluate $\overline{z - \overline{z}}$.
20. Find all the complex numbers z (and illustrate them on the complex plane) for which we have:
$$|\operatorname{Im} z| > 2.$$
21. Find all the complex numbers z (and illustrate them) for which we have:
$$|\operatorname{Im} z + 1| < 1 \ \ \wedge |\operatorname{Re} z - 1| \leq 2.$$
22. Solve the following equation on the set of the complex numbers:
$$x^2 + 2x + 3 = 0.$$
23. Solve the following equation on the set of the complex numbers:
$$x^3 + 1 = 0.$$
24. Solve the following equation on the set of the complex numbers:
$$x^2 + 8 = 0.$$
25. Solve the following equation on the set of the complex numbers:
$$x^4 - 16 = 0.$$
26. Solve the following equation on the set of the complex numbers:
$$x^4 + x^2 - 1 = 0.$$

9.2. Exercises

9.2.1. Exercises for class work

1. Find all the numbers $x, y \in \mathbb{R}$ for which the following equalities are true (i is the imaginary unit : $i^2 = -1$):

(a) $(1-2i)\cdot x + (1+2i)\cdot y = 1+i$;

(b) $(2+i)\cdot x - (2-i)\cdot y = x - y + 2i$;

(c) $(4-3i)\cdot x^2 + (3+2i)\cdot xy = 4y^2 - \frac{1}{2}x^2 + (3xy - 2y^2)\cdot i$;

(d) $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$.

2. Evaluate the following complex numbers and give their algebric form:

(a) $\dfrac{1}{2-3i}$;

(b) $\dfrac{1+5i}{3+2i}$;

(c) $(1-2i)\cdot(5+i)$;

(d) $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+i}}}$;

(e) $(2-i)^2 + (2+i)^3$;

(f) $(3-\sqrt{2}i)^3 \cdot (3+\sqrt{2}i)$;

(g) $\dfrac{1+i}{3-i} + \dfrac{3-i}{1+i}$;

(h) $\dfrac{(1+i)^2}{1-i} + \dfrac{(1-i)^3}{(1+i)^2}$;

(i) $\left(\dfrac{1+i}{1-i}\right)^{2018} + \left(\dfrac{1-i}{1+i}\right)^{2019}$.

3. Prove that the given complex numbers are roots of the following equations:

(a) $x^3 - x^2 + 8x + 10 = 0$; $z_1 := 1+3i \;\wedge\; z_2 := 1-3i$;

(b) $x^4 + x^2 + 1 = 0$; $z_1 := -\dfrac{1}{2} + i\cdot\dfrac{\sqrt{3}}{2} \;\wedge\; z_2 := -\dfrac{1}{2} - i\cdot\dfrac{\sqrt{3}}{2}$.

4. Consider the following complex numbers:

$$z_1 = \frac{\sqrt{2}+i}{\sqrt{2}-i} \quad \wedge \quad z_2 = \frac{i}{-2\sqrt{2}+2i}.$$

Evaluate:

$$z_1 + z_2; \;\; z_1 \cdot z_2; \;\; \frac{z_1}{z_2}; \;\; z_1^2 + z_2^2; \;\; z_1^3 + z_2^3; \;\; z_1^4 + z_2^4; \;\; z_1^{-3} \cdot z_2^5.$$

5. Evaluate:

(a) $S_{2018} = \sum_{k=0}^{2018} i^k$;

(b) $S_{2019} = \sum_{k=0}^{2019} (-1)^k \cdot i^k$.

6. Suppose that for the complex numbers $z, w \in \mathbb{C}$ we have: $|z| = |w| = 1$. Prove that in this case:

$$\frac{z+w}{1+z\cdot w} \in \mathbb{R}.$$

7. Evaluate $|z|$, if:

$$z = \left(\sqrt{2+\sqrt{2}} + i \cdot \sqrt{2-\sqrt{2}}\right)^4.$$

8. Find all those complex numbers z for which z^3 is an imaginary number.

9. Which complex numbers $z \in \mathbb{C}$ satisfy the following equations:

(a) $z^2 = 1 + i$;

(b) $z^3 = \overline{z}$?

Illustrate these numbers on the complex plane.

10. Prove that for all complex numbers $z, w \in \mathbb{C}$ we have:

$$|z+w|^2 + |z-w|^2 = 2 \cdot (|z|^2 + |w|^2).$$

What is the geometric meaning of this equation?

11. What is the smallest and the greatest value of $|z - i - 1|$ and at which values of z will it take them, if:

(a) $\operatorname{Im} z = -2$?

(b) $\operatorname{Im} z = 2 \cdot \operatorname{Re} z$?

(c) $|z+1| = 1$?

Polynomials with real coefficients

12. Solve the following equations on $\mathbb{C}$:

(a) $x^3 - 1 = 0$;

(b) $x^4 - 1 = 0$;

(c) $x^2 - 2\sqrt{2} \cdot x + 5 = 0$;

(d) $x^3 - 9x^2 + 18x + 28 = 0$;

(e) $x^4 - 30x^2 + 289 = 0$.

9.2.2. Homework and more exercises to practice

1. Find $x, y \in \mathbb{R}$ so that:

(a) $2x + i + x \cdot i \cdot (x+y) = 3 + y \cdot (1-i)$;

(b) $3 \cdot \sqrt{x^2 - 2y} + (1-i)x^2 = 2 \cdot (1+2i)y + 4 - 19i$;

(c) $\dfrac{2}{3x+iy} - \dfrac{3}{5+i} = 2$;

(d) $\dfrac{x-1}{1+i} + \dfrac{y+1}{1-i} = \dfrac{i}{2}$.

2. Evaluate the following expressions, and give their algebraic form:

(a) $\dfrac{3}{\sqrt{2}-i}$;

(b) $\dfrac{i-\sqrt{5}}{i+\sqrt{5}}$;

(c) $(i^2 - 7i) \cdot (2+i)$;

(d) $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{i-1}}}$;

(e) $(2+i)^2 + (2-3i)^3$;

(f) $\dfrac{i-6}{2+5i} + \dfrac{2-i}{2+i}$;

(g) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^6 + \left(\dfrac{-1-i\sqrt{3}}{2}\right)^6$;

(h) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^5 + \left(\dfrac{-1-i\sqrt{3}}{2}\right)^5$;

(i) $\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^6$;

(j) $\dfrac{(1/\sqrt{2}+i)^3-(1/\sqrt{2}-i)^3}{(1/\sqrt{2}+i)^2-(1/\sqrt{2}-i)^2}$;

(k) $\left(\dfrac{19+7i}{9-i}\right)^4 + \left(\dfrac{20+5i}{7+6i}\right)^4$.

3. Prove that the given numbers are the roots of the correspondent equation:

(a) $x^3-3\sqrt{2}\cdot x^2+7x-3\sqrt{2}=0$; $z_1:=\sqrt{2}-i \ \wedge \ z_2:=\sqrt{2}+i$;

(b) $x^4-2x^3+(6+\sqrt{2})x^2-8x+4\cdot(\sqrt{2}+2)=0$; $z_{1,2}:=\pm 2i \ \wedge \ z_{3,4}:=1\pm i\cdot\sqrt{1+\sqrt{2}}$.

4. Consider the following complex numbers:

$$z_1=\frac{5i}{\sqrt{3}+\sqrt{2}\cdot i} \ \wedge \ z_2=\frac{5}{\sqrt{2}+\sqrt{3}\cdot i}.$$

Evaluate the following expressions and give their algebraic form:

$$z_1+z_2;\ z_1\cdot z_2;\ \frac{z_1}{z_2};\ z_1^2+z_2^2;\ z_1^3+z_2^3;\ z_1^4+z_2^4;\ z_1^3\cdot z_2^{-3}-z_1^{-3}\cdot z_2^3\,.$$

5. Find all the numbers $n\in\mathbb{N}^+$ for which we have:

$$(1+i)^n=(1-i)^n\,.$$

6. Prove that for all natural numbers $n\in\mathbb{N}$ we have:

$$(1+i)^n+(1-i)^n\in\mathbb{R}\,.$$

7. Evaluate:

(a) $S_{2018}=\displaystyle\sum_{k=1}^{2018}(1+i)^k$;

(b) $S_{2019}=\displaystyle\sum_{k=1}^{2019}\left(\frac{1-i}{1+i}\right)^{3k}$.

8. Prove that for all the parameters $a \in \mathbb{R}$ the absolute value of the following complex number is 1:

$$z := \frac{1+ia}{1-ia}.$$

9. Find $|z|$, if:

$$z = \frac{1}{\left(\sqrt{\sqrt{2}+1} - i \cdot \sqrt{\sqrt{2}-1}\right)^4}.$$

10. Find all the complex numbers z for which z^4 is a real number. Illustrate these numbers on the complex plane.

11. Find all the complex numbers z for which z^4 is an imaginary number. Illustrate them on the complex plane.

12. Find all the complex numbers z for which we have:

(a) $|z+i| = |\overline{z}-1| = |z-iz|$;

(b) $|z+1-i| = 1 \ \wedge \ |z-1+i| = \sqrt{5}$;

(c) $|z+2+i| = 1 \ \wedge \ |z| = \sqrt{5}-1$;

(d) $|z-1| = |1+iz| = \sqrt{z \cdot \overline{z}}$.

13. What complex numbers $z \in \mathbb{C}$ satisfy the following conditions:

(a) $z^2 = 1 - i\sqrt{3}$;

(b) $z^3 = \dfrac{i}{\overline{z}}$?

Illustrate these numbers on the complex plane.

14. What will be the smallest and the greatest value of $|z-1+i|$ (and for what values of z does it take them), if:

(a) $\operatorname{Im} z = 2$;

(b) $\operatorname{Re} z = -1$;

(c) $|z-i| = 1$.

Polynomials with real coefficients

15. Solve the following equations on $\mathbb{C}$:

(a) $x^2 + 2x + 3 = 0$;

(b) $x^3 + 1 = 0$;

(c) $x^6 + 1 = 0$;

(d) $x^3 - \sqrt{2} \cdot x^2 + 4x - 4\sqrt{2} = 0$;

(e) $x^4 - 4x^3 + 10x^2 - 20x + 25 = 0$.